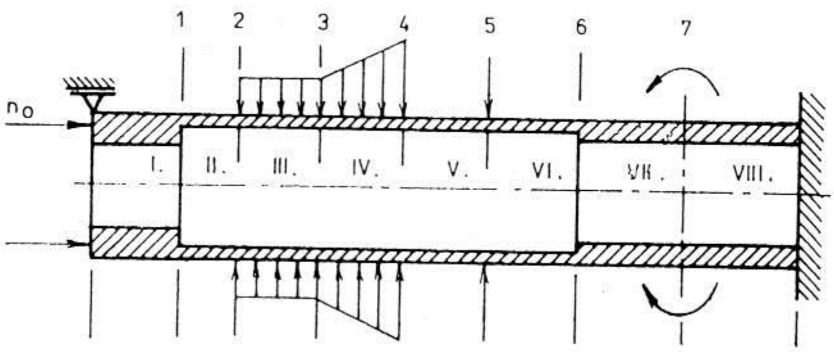
Cylindrical momentum shell

The **momentum theory** of **cylindrical shell** should be applied if the assumptions of the membrane stress state in the shell are disturbed; this occurs under the following conditions (see the figure below):

1. points of the middle surface are joined with another body (radial displacements or/and rotations are constrained, i.e.),
2. stepwise change in the shell stiffness (i.e. in its wall thickness, radius or modulus of elasticity of the material),
3. discontinuity in loads (line force, line moment) or their derivatives (change in the pressure distribution – see fig. below).

Stepwise changes in the slope or curvature of the middle surface cannot occur at a cylindrical shell.



If the conditions of the membrane shell theory are disturbed, the factor of safety may decrease by a factor ranging from 1 to 2 (in comparison with that calculated from the membrane shell theory).

Momentum shell is a thin-wall body not meeting the assumptions of the membrane stress state. Cylindrical shell is a special case of axisymmetric shell, therefore it must meet the conditions of axisymmetry in geometry, material properties, supports and loads, otherwise the displacements, strains and stresses cannot be axisymmetric. **Radial displacement *u*** represents a major deformation parameter, **axial displacement *w*** is an independent deformation parameter as well; ***υ=tgυ=du/dz*** is introduced as a minor (slave) deformation parameter and represents the **angle of rotation** of a tangent line to the meridian caused by the shell deformation.

Stress tensor:

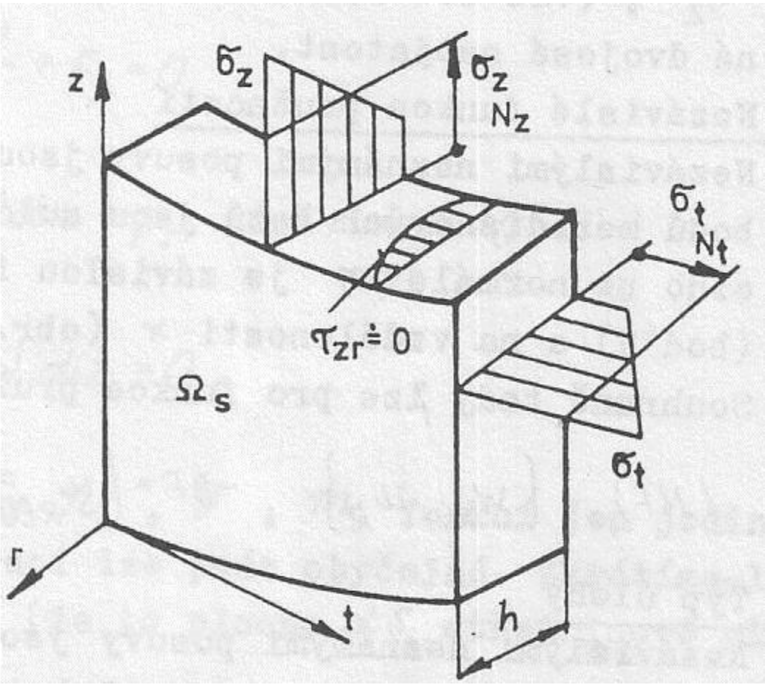
- **circumferential stress** σ_t represents the highest principal stress in most cases,
- at a cylinder the meridian line is parallel to the z axis, therefore the meridian stress can be denoted as **axial stress σ_z**,
- radial stress σ_r is negligible (in contrast to a thick wall body) in consequence of the small shell thickness – similar to the membrane shell theory,
- **shear stress** τ_{rz} is non-zero and must be taken into consideration in the equations of equilibrium; however, similarly to the bended long slender beams and Kirchhoff's plates, its magnitude is not significant for evaluation of failure risk.

The stress tensor can be therefore written in the following matrix form:

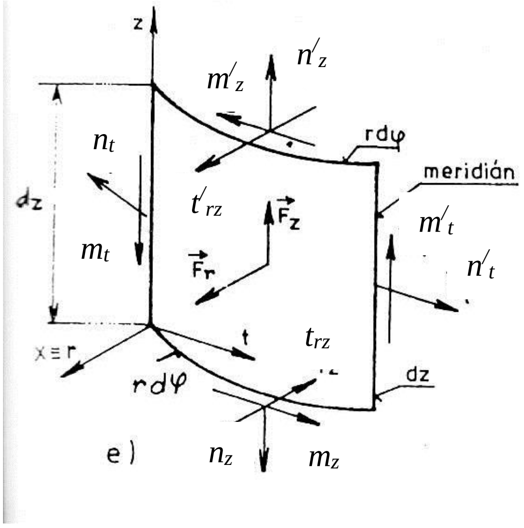
$$T_\sigma = \begin{bmatrix} \sigma_r \cong 0 & 0 & \tau_{zr} \approx 0 \\ 0 & \sigma_t & 0 \\ \tau_{rz} \approx 0 & 0 & \sigma_z \end{bmatrix}$$

Only axisymmetric (i.e. distributed) loads are acceptable. They can be induced either by pressure p_r of the fluid medium inside or outside the shell or by fixation of the shell (typically when a flange is fixed to the shell). In this way not only distributed forces (along the circumference) but also distributed moments can be induced; this occurs typically if the shell is supported (in vertical position) by means of the flange. If this flange is fixed to the shell by interference only, then (in addition to the radial contact pressure p_r acting at the interface) also axial component p_z of the pressure can be induced by friction equilibrating gravitational forces. In the other cases the axial component of pressure p_z is negligible.

Stress distribution in an element of cylindrical momentum shell



Force and couple resultants according the figure can be introduced using the following equations of **static equivalence**:



$$n_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z dx \; [N/mm]$$

$$n_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_t dx \; [N/mm] \qquad (1a)$$

$$t_{rz} = t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{rz} dx \; [N/mm] \qquad (1b)$$

$$m_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} x\sigma_z dx \qquad m_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} x\sigma_t dx \; [Nmm/mm] \qquad (1c)$$

Relations between the distributed resultants in opposite sections are as follows:

$n_z^/ = n_z + dn_z \quad ; \qquad n_t^/ = n_t$

$t_{rz}^/ = t_{rz} + dt_{rz} \Rightarrow t^/ = t + dt$ (subscripts are not necessary and thus not used below)

$m_z^/ = m_z + dm_z \quad ; \qquad m_t^/ = m_t$

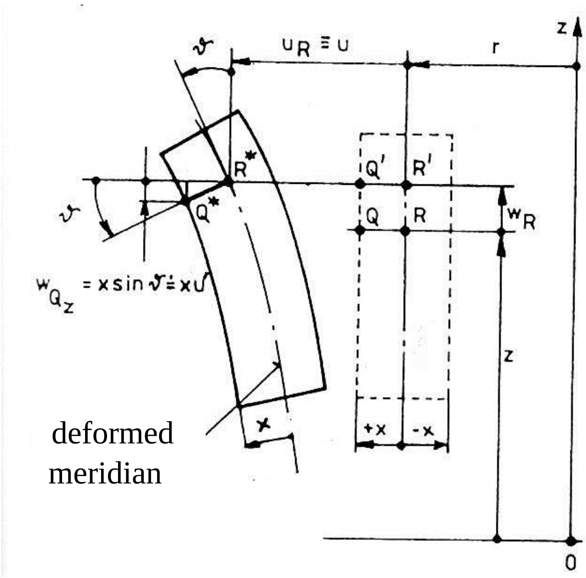
Three applicable equations can then be obtained on the basis of static equilibrium of the element:

$$\sum F_z : \frac{dn_z}{dz} + p_z = 0 \qquad (2a)$$

$$\sum F_r : \frac{dt}{dz} - \frac{n_t}{r} + p_r = 0 \qquad (2b)$$

$$\sum M_t : -\frac{dm_z}{dz} + t = 0 \qquad (2c)$$

strain-displacement equations



$$w = w_R - w_{Qz} = \\ = w_R - x.\sin\upsilon = \\ = w_R - x.\frac{du}{dz} \qquad (3a)$$

$$\varepsilon_z = \frac{dw}{dz} = \frac{dw_R}{dz} - x.\frac{d^2u}{dz^2} \qquad (3b)$$

$$\varepsilon_r = \frac{du}{dr}$$

$$\varepsilon_t = \frac{u}{r}$$

Constitutive equations can be applied for plane stress conditions in the form:

$$\sigma_z = \frac{E}{1-\mu^2}\left[\varepsilon_z + \mu\varepsilon_t\right] \qquad (4a)$$

$$\sigma_t = \frac{E}{1-\mu^2}\left[\varepsilon_t + \mu\varepsilon_z\right] \qquad (4b)$$

By substituting constitutive (4) and strain-displacement (3) equations into equations (1) we can obtain after some manipulations:

$$m_z = -\frac{Eh^3}{12\left(1-\mu^2\right)}\frac{d^2u}{dz^2} = -B\frac{d^2u}{dz^2} \qquad (5a)$$

$$m_t = -\mu B\frac{d^2u}{dz^2} = \mu m_z \qquad (5b)$$

In these equations the **bending stiffness B** of the shell was introduced

$$B = \frac{Eh^3}{12\left(1-\mu^2\right)} \qquad (6)$$

By substituting eq. (5a) into (2c) we obtain for shear force (per unit length) t:

$$t = -B\frac{d^3u}{dz^3} \tag{6a}$$

This result will be substituted into eq. (2b).

If we substitute strain-displacement eq. (3b) into Hooke's law eqs.(4a) and (4b) and subsequently into (1a), we obtain the following two formulas:

$$n_z = \frac{Eh}{1-\mu^2}\left[\frac{dw}{dz} + \mu\frac{u}{r}\right] \quad ; \quad n_t = \frac{Eh}{1-\mu^2}\left[\frac{u}{r} + \mu\frac{dw}{dz}\right]$$

After elimination of *dw/dz* from these two equations we obtain the formula for n_t depending only on the radial component of displacement *u*:

$$n_t = \mu n_z + Eh\frac{u}{r} \tag{6b}$$

By substituting this result into eq. (2b) we obtain the following 4th order differential equation:

$$\frac{d^4u}{dz^4} + 4\beta^4 u = f(p_r, p_z)$$

$$\frac{d^4u}{dz^4} + 4\beta^4 u = \frac{1}{B}\left[p_r - \frac{\mu}{r}\left(c_0 - \int p_z(z)dz\right)\right], \tag{7}$$

where eq. (2a) was also applied for n_z and the parameter β was introduced as follows:

$$\beta = \sqrt[4]{\frac{3(1-\mu^2)}{r^2h^2}} \quad \left[m^{-1}\right] \tag{8}$$

The solution to the homogeneous part of equation (7) is known in the following form

$$u_{\text{hom}} = e^{-\beta z}\left(c_1 \sin\beta z + c_2 \cos\beta z\right) + e^{\beta z}\left(c_3 \sin\beta z + c_4 \cos\beta z\right) \tag{9}$$

For this 4th order differential equation, **4 boundary conditions** are required; they are formulated typically for radial displacements ***u*** and their derivatives ***υ*** at both ends of the shell. As the influence of the constrained sections decreases steeply (exponentially) with the axial distance from them and both solutions tend to membrane stresses, all the constrained sections of the shell can be solved separately if their distance is sufficiently large.

In this case only one section of the shell is constrained in its radial displacements and integration **constants c_3 and c_4 equal zero**. The impact of the constrained section is negligible when the momentum stress becomes lower than 2% of membrane stress; it can be shown that this distance is approximately ***$l_0=4/\beta$*** and behind this value the membrane shell theory is valid. Consequently, if the distance between two locations of membrane state violation is **higher than *$2l_0$***, they do not influence each other (so called „**long shell**"), and only two boundary conditions (both for *$z=0$*) are needed.
Thus the first step in application of momentum shell theory is decision whether the criterion of long shell is met; for this purpose, we use the formula:

$$l_0 \cong \frac{4}{\beta} \tag{10a}$$

For steels (μ=0,3) this formula can be simplified into the form:

$$l_0 \cong 3\sqrt{rh} \tag{10b}$$

As the resulting l_o is typically smaller than the radius, $2l_o$ is smaller than the shell diameter.
In most practical applications, another simplification is possible. If the radial component of pressure is constant along the solved part of the shell and we can neglect the axial pressure component p_z, then it holds (from eq. (2a)) $n_z = const.$ and the differential equation has the following simplified form:

$$\frac{d^4u}{dz^4} + 4\beta^4 u = \frac{1}{B}\left[p_r - \frac{\mu}{r}n_z\right] \tag{11}$$

with the solution consisting of its homogeneous and particular solutions

$$u = u_{\hom} + u_{part}$$

$$u_{\hom} = e^{-\beta z}\left(c_1 \sin\beta z + c_2 \cos\beta z\right)$$

$$u_{part} = \frac{r^2}{Eh}\left[p_r - \frac{\mu}{r}n_z\right]$$

$$u = e^{-\beta z}\left(c_1 \sin\beta z + c_2 \cos\beta z\right) + \frac{r^2}{Eh}\left[p - \frac{\mu}{r}n_z\right] \tag{12}$$

which represents an applicable simplification of the more general equation below valid for a long shell when $p_z = 0$ (and thus $p_r = p$):

$$u = e^{-\beta z}\left(c_1 \sin\beta z + c_2 \cos\beta z\right) + e^{\beta z}\left(c_3 \sin\beta z + c_4 \cos\beta z\right) + \frac{r^2}{Eh}\left[p_r - \frac{\mu}{r}\left(c_0 - \int p_z(z)dz\right)\right]$$

Boundary conditions can be formulated on the basis of constraints of deformation parameters (displacement and angle of rotation) and/or external loads (distributed line forces or couples). Their formulation can be as follows:

Pin support: for $z=0$ it holds $u = 0$.

Fixed support: for $z=0$ it holds $u=0,\ \upsilon = \frac{du}{dz} = 0$.

Free edge (not fixed and unloaded): for $z=0$ (or $z=l$) it holds

$$m_z = -B\frac{d^2u}{dz^2} = 0, \qquad t = -B\frac{d^3u}{dz^3} = 0,$$

For an **edge loaded** by a distributed force or couple, similar boundary conditions with non-zero values of these quantities can be formulated. This external loads are taken as positive if they bend the shell inwards.

***Note**: If the axial component of pressure is not negligible, then the integration constant c_0 in eq. (7) equals the magnitude of axial distributed force n_z in the critical location (for z=0).*

Procedure of solution to a direct problem:

1. On the basis of distance l between the neighbouring violations of the membrane stress state, we decide whether the shell can be solved as a „long shell" (if $l>2l_0$). In this case the origin of the coordinate system ($z=0$) is located in the centre of the section where membrane assumptions are violated.

2. We create free body diagram of a finite element of the shell and express explicitly the axial distributed load n_z from its equation of static equilibrium of forces acting in z direction.

3. We substitute n_z into eq. (12) and formulate two boundary conditions (both in the critical location, i.e. for $z=0$). In addition to radial displacement u, these boundary conditions can be formulated for its derivatives ($\upsilon=du/dz;\ m_z;\ t$).

4. We determine the integration constants c_1 and c_2 in eq. (12); thus the equation for displacements is obtained.

5. We calculate the distributed loads m_z, m_t, n_t as functions of z coordinate and find their maximum values (mostly in $z=0$). Here we apply eqs. (5a), (5b) for moments; the distributed force in tangential direction can be calculated using

$$n_t = \mu n_z + Eh\frac{u}{r}$$

6. Extreme stresses in the dangerous points are given by the following formulas

$$\sigma_{t\max} = \frac{n_t}{h} \pm \frac{6m_t}{h^2} \quad \text{and} \quad \sigma_{z\max} = \frac{n_z}{h} \pm \frac{6m_z}{h^2} \quad ,$$

   where members with n and m relate to the membrane and bending components of stresses, respectively.

7. As the third principal stress σ_r equals zero, the reduced Tresca stress (valid for a ductile material) equals to the largest absolute value of the above extreme stresses. This value is then used for calculation of the factor of safety. The same formulas hold also for brittle materials, but for negative membrane stresses (they occur under outer pressure) the sign of the resulting stresses must be taken into consideration due to different ultimate values in tension and compression.