INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Brittle fracture

of a body with an initial crack

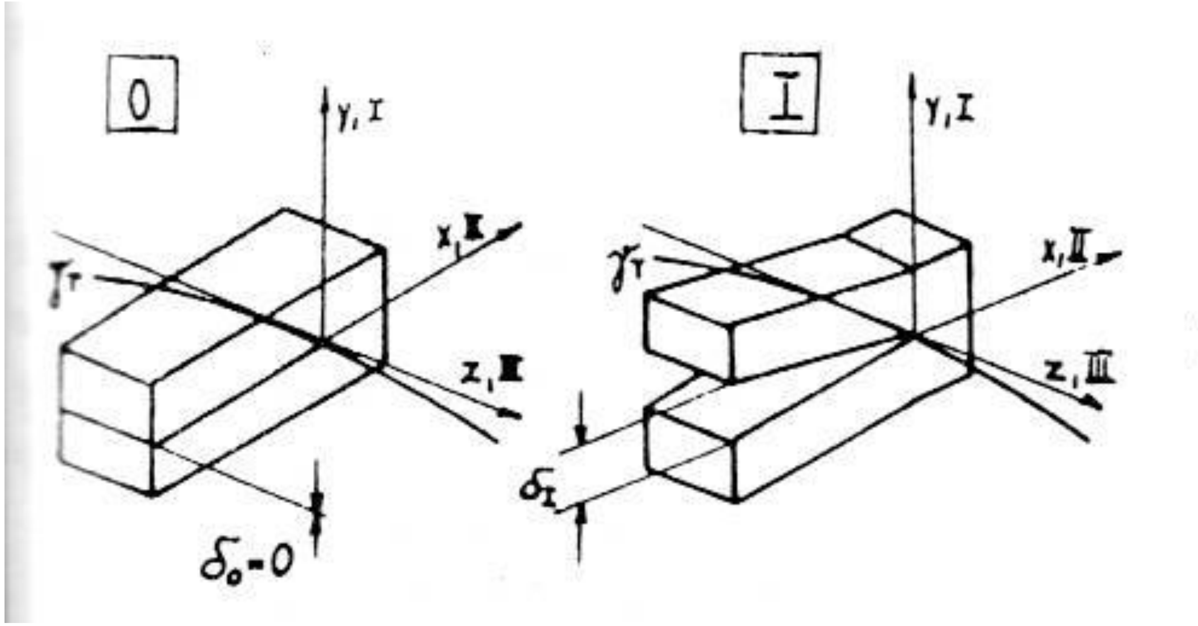
LEFM - linear elastic fracture mechanics

Basic notions:	Phases of the fracture process:
Crack Nucleation (initiation) site Crack front Crack tip Fracture area	1. Nucleation phase Initiation (detection) of a macroscopic crack 2. Stable crack propagation Critical crack length, failure by a leakage 3. Unstable crack propagation Fracture (fatigue)

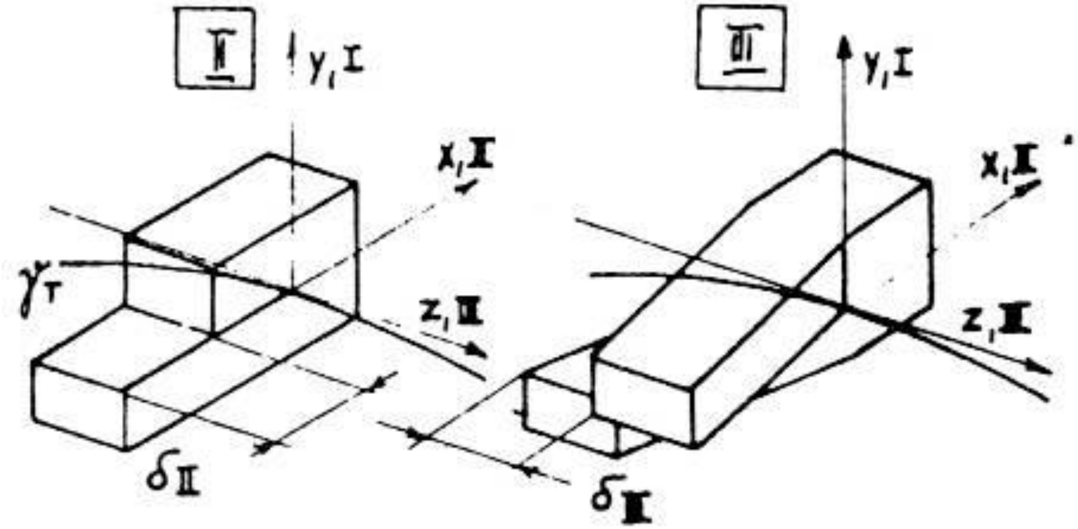
Discontinuity vector δ

characterizing deformation in the crack front surroundings

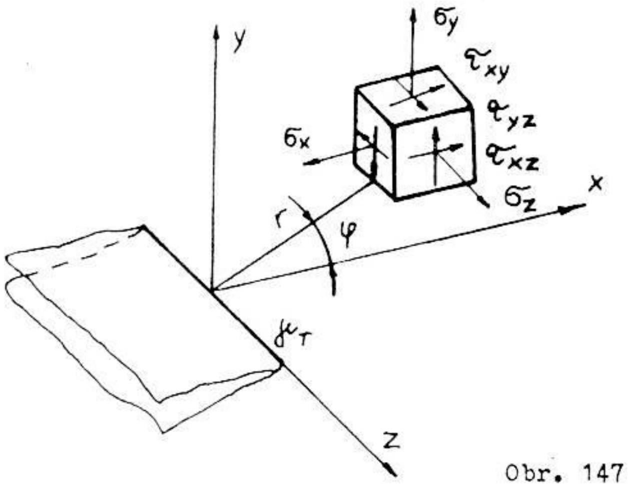
Zero opening **MODE I (opening)**



MODE II (in-plane shear) **MODE III** (out-of-plane shear)



Linear elastic fracture mechanics (LEFM)



Mode I:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\varphi}{2} \begin{bmatrix} 1 - \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2} \\ 1 + \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2} \\ \sin\frac{\varphi}{2}\cos\frac{3\varphi}{2} \end{bmatrix} \tag{1}$$

plane strain (w=0):

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{2K_I(1+\mu)}{E}\sqrt{\frac{r}{2\pi}} \begin{bmatrix} \cos\frac{\varphi}{2}\left(1 - 2\mu + \sin^2\frac{\varphi}{2}\right) \\ \sin\frac{\varphi}{2}\left(2 - 2\mu - \cos^2\frac{\varphi}{2}\right) \end{bmatrix} \tag{2}$$

Mode II: $$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\varphi}{2}\begin{bmatrix}-\sin\frac{\varphi}{2}\left(2+\cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}\right)\\ \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}\\ \cos\frac{\varphi}{2}\left[1-\sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}\right]\end{bmatrix} \quad (3)$$

$\tau_{xz}=\tau_{yz}=0$; $\sigma_z=0$ for plane stress

$\sigma_z=\mu\left(\sigma_x+\sigma_y\right)$ for plane strain

$$\begin{bmatrix}u\\ v\end{bmatrix}=\frac{2K_{II}(1+\mu)}{E}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\sin\frac{\varphi}{2}\left(2-2\mu+\cos^2\frac{\varphi}{2}\right)\\ \cos\frac{\varphi}{2}\left(1-2\mu+\sin^2\frac{\varphi}{2}\right)\end{bmatrix} \quad (4)$$

Mode III: $$\begin{bmatrix}\tau_{xz}\\ \tau_{yz}\end{bmatrix}=\frac{K_{III}}{\sqrt{2\pi r}}\begin{bmatrix}-\sin\frac{\varphi}{2}\\ \cos\frac{\varphi}{2}\end{bmatrix} \quad (5)$$

u=v=0; $$w=\frac{2K_{III}(1+\mu)}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\varphi}{2} \quad (6)$$

Generalization of the equations

$$\sigma_{ij} = \frac{K_e}{\sqrt{2\pi r}} f_{ij}^{e}(\varphi) \quad (7)$$

$$u_i = \frac{2K_e(1+\mu)}{E}\sqrt{\frac{r}{2\pi}}\, g_i^{e}(\quad (8)$$

For a general mode (general orientation of the discontinuity vector δ) the principle of superposition can be applied with the following resulting formulas:

$$\sigma_{ij} = \frac{1}{\sqrt{2\pi r}}\left[K_I . f_{ij}^{I}(\varphi) + K_{II} . f_{ij}^{II}(\varphi) + K_{III} . f_{ij}^{III}(\varphi)\right] \quad (9)$$

$$u_i = \frac{2(1+\mu)}{E}\sqrt{\frac{r}{2\pi}}\left[K_I . g_i^{I}(\varphi,\mu) + K_{II} . g_i^{II}(\varphi,\mu) + K_{III} . g_i^{III}(\varphi,\mu)\right] \quad (10)$$

K-concept

is based on the basic quantity for description of the crack behaviour called **stress intensity factor** (K-factor)

K-factor can determine

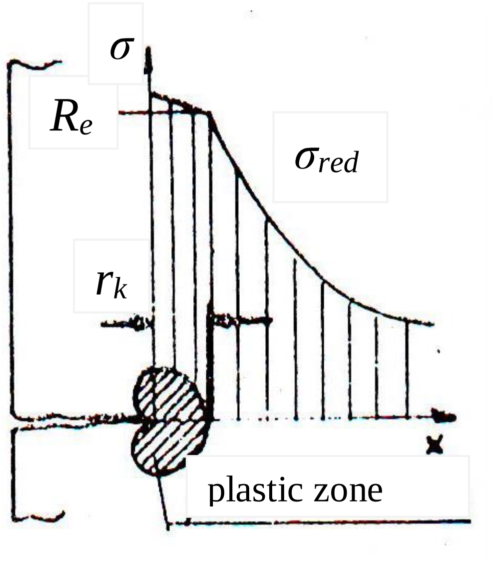
1. Dimensions of the plastic zone
2. Shape of the opened crack
3. Crack driving force — strain energy released by the crack propagation

Dimensions of the plastic zone

$$r_k = \frac{\alpha}{\pi}\left(\frac{K_I}{R_e}\right)^2 \qquad (11)$$

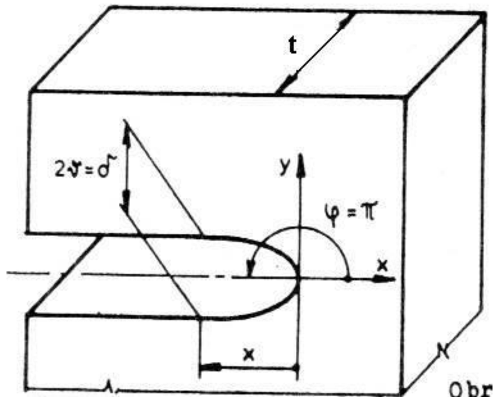
α=1 for thickness t <10mm
(plane stress, accurate for t<2mm,
for higher thickness conservative)

α=1/3 for thickness t >10mm
(plane strain)



Shape of the opened crack (for mode I)

Given by the lower one of eqs. (2) after substituting φ=π and consequently r=x.



$$\frac{\delta}{2} = v = \frac{2K_I(1+\mu)}{E}\sqrt{\frac{x}{2\pi}}\sin\frac{\varphi}{2}\left(2-2\mu-\cos^2\frac{\varphi}{2}\right) = \frac{2K_I(1+\mu)}{E}\sqrt{\frac{x}{2\pi}}\left(2-2\mu\right)$$

Strain energy released by the crack propagation

$$\frac{dW_e}{t.da} = -\frac{K_I^2}{E} < 0 \quad \left[\frac{Nm}{m^2} = \frac{N}{m} \right] \tag{12}$$

The energy released by a unit increase of the fracture area is also called crack driving force G

$$G = \left| \frac{dW_e}{t.da} \right| = \frac{K_I^2}{E} \tag{13}$$

It holds for balance of energy increments during the crack propagation

$$dW_p = \left| dW_e \right| + dW_l \tag{14}$$

where

dW_p - represents the energy increment needed for the crack propagation,

$/dW_e/$ - represents the decrease of elastic energy due to the crack propagation,

dW_l - represents the work done by external loads.

On the basis of this balance, the following three situations can be distinguished:

1. $dW_p > \left| dW_e \right| \Rightarrow dW_l > 0$ **stable crack propagation**

An additional work is required for crack propagation, to be done by external loads.

2. $dW_p = \left| dW_e \right| \Rightarrow dW_l = 0$ **unstable crack propagation**

The crack starts to propagate without any work to be done by external loads.

3. $dW_p < \left| dW_e \right| \Rightarrow dW_l < 0$ **unstable crack propagation with an excess of energy**

The crack propagates without any work to be done by external loads; moreover, an excess of elastic strain energy is released and transformed e.g. into kinetic energy of fragments.

The energy increment needed for crack propagation can be expressed per unit increase of fracture area as follows:

$$\lambda_p = \frac{dW_p}{t.da} \qquad (15)$$

So we obtained a specific material characteristic λ_p which, however, cannot be measured experimentally. However, equality of both energies dW_p and $/dW_e/$ holds for the limit state, thus we can replace them in eq. (13). The limit of unstable crack propagation can then be expressed as follows:

$$dW_p = \lambda_p.tda = \frac{K_I^2}{E}.tda \quad => \quad K_I = \sqrt{\lambda_p E} \quad \left[MPa\sqrt{m}\right] \qquad (16)$$

In this way we have transformed the energy needed for creation of a unit fracture area into another quantity being decisive for unstable crack propagation **— stress intensity factor**. Its value under which unstable crack propagation occurs is denoted as K_{IC} and called **fracture toughness**. This is a material characteristic that can be evaluated experimentally. It is done mostly by using three-point bending of notched and cracked specimens.

Consequently, the criterion of unstable crack propagation can be written in the following simple form:

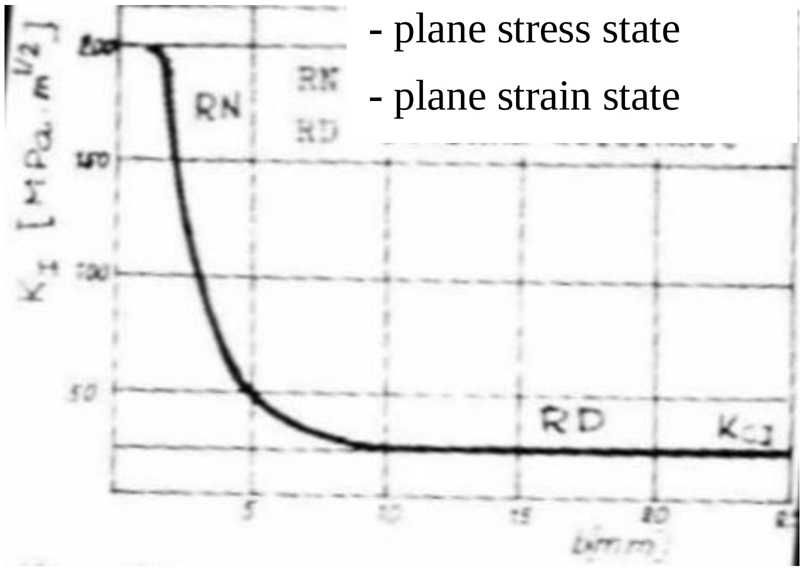
$$K_I = K_{IC}$$

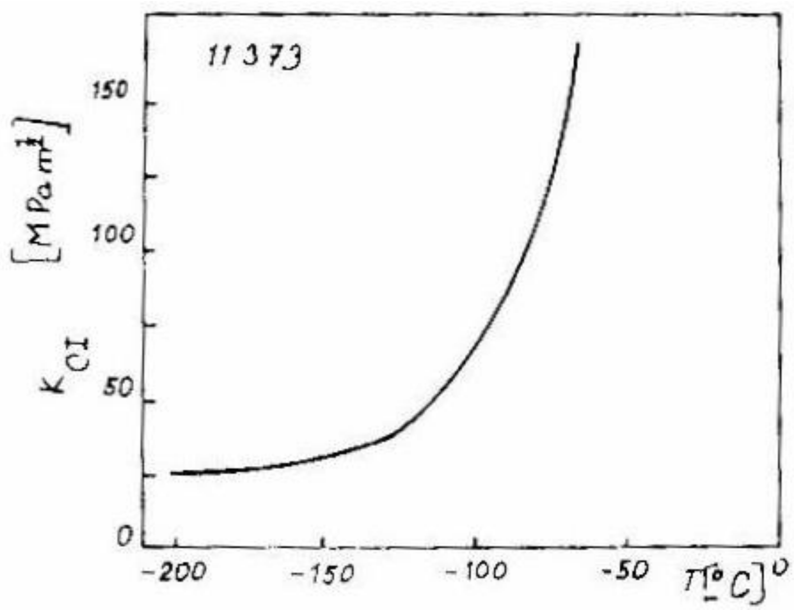
Stable crack propagation occurs if

$$K_I < K_{IC}$$

Dependence of the ultimate K-factor value on the component thickness (type of stress state)

Example of the dependence of fracture toughness on temperature



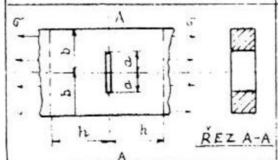
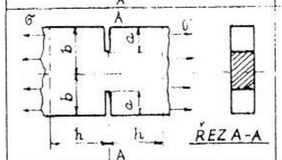
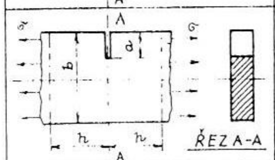
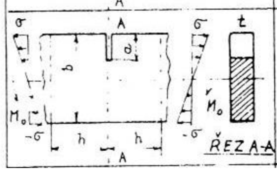
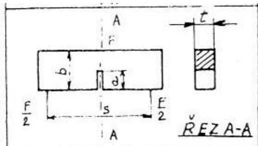
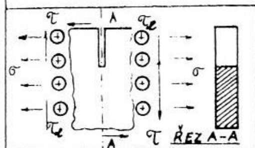
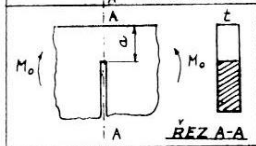
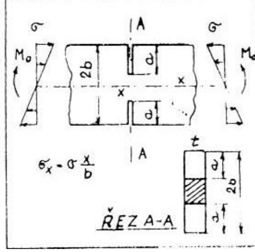


K_{CI}=K_{IC} = fracture toughness

Definition:

Fracture toughness is the value of stress intensity factor under which unstable crack propagation starts under conditions of plain strain and small plastic zone in the vicinity of the crack front.

Examples of formulas for K-factor calculation

SCHEMA	VZTAH PRO K	S PŘESNOSTÍ PLATÍ PRO
	$K_I = \sigma\sqrt{\pi a}\,\dfrac{1-0{,}5\left(\frac{a}{b}\right)+0{,}37\left(\frac{a}{b}\right)^2-0{,}044\left(\frac{a}{b}\right)^3}{\sqrt{1-\frac{a}{b}}}$ $K_{II} = K_{III} = 0$	0,3% PRO JAKÉKOLIV $\frac{a}{b}$ $a < b$ $\frac{h}{b} \geq 3$
	$K_I = \sigma\sqrt{\pi a}\,\dfrac{1{,}122-0{,}561\left(\frac{a}{b}\right)-0{,}205\left(\frac{a}{b}\right)^2+0{,}471\left(\frac{a}{b}\right)^3-0{,}19\left(\frac{a}{b}\right)^4}{\sqrt{1-\frac{a}{b}}}$ $K_{II} = K_{III} = 0$	0,5% PRO JAKÉKOLIV $\frac{a}{b}$ $a < b$ $\frac{h}{b} \geq 2{,}75$
	$K_I = \sigma\sqrt{\pi a}\left[1{,}12-0{,}231\left(\frac{a}{b}\right)+10{,}55\left(\frac{a}{b}\right)^2-21{,}72\left(\frac{a}{b}\right)^3+30{,}39\left(\frac{a}{b}\right)^4\right]$ $K_{II} = K_{III} = 0$	0,5% PRO $\frac{a}{b} \leq 0{,}6$ $a < b$ $\frac{h}{b} \geq 1$
	$\sigma = \frac{6M_o}{tb^2}$ $K_I = \sigma\sqrt{\pi a}\left[1{,}122-1{,}4\left(\frac{a}{b}\right)+7{,}33\left(\frac{a}{b}\right)^2-13{,}08\left(\frac{a}{b}\right)^3+14{,}0\left(\frac{a}{b}\right)^4\right]$ $K_{II} = K_{III} = 0$	0,2% PRO $\frac{a}{b} \leq 0{,}5$ $a < b$ $\frac{h}{b} \geq 2$
	$\sigma = \frac{6M_o}{tb^2} \quad \left(M_o = \frac{Fs}{4}\right)$ $K_I = \sigma\sqrt{\pi a}\left[1{,}107-2{,}12\left(\frac{a}{b}\right)+7{,}71\left(\frac{a}{b}\right)^2-13{,}55\left(\frac{a}{b}\right)^3+14{,}25\left(\frac{a}{b}\right)^4\right]$ $K_{II} = K_{III} = 0$	0,2% PRO $\frac{a}{b} \leq 0{,}6$ $a < b$ $\frac{s}{b} = 8$
	$K_I = 1{,}1215\,\sigma\sqrt{\pi a}$ $K_{II} = 1{,}1215\,\tau\sqrt{\pi a}$ $K_{III} = \tau_\ell\sqrt{\pi a}$	K_I, K_{II} PRAKTICKY PŘESNÉ K_{II} PŘESNĚ
	$K_I = 3{,}975\,\frac{M_o}{ta\sqrt{a}}$ $K_{II} = K_{III} = 0$	DO 0,1%
	$\sigma = \frac{3M_o}{2tb^2} \qquad K_{II} = K_{III} = 0$ $\sigma_N = \frac{3M_o}{t2(b-a)^2} = \frac{\sigma}{1-\left(\frac{a}{b}\right)^2}$ $K_I = \sigma\sqrt{\pi a}\cdot f_1\left(\frac{a}{b}\right) = \sigma_N\sqrt{\pi a}\,f_2\left(\frac{a}{b}\right)$ $f_1\left(\frac{a}{b}\right) = \dfrac{\frac{4}{3\pi}\left\{1+\frac{1}{2}\left(\frac{a}{b}\right)+\frac{3}{8}\left(\frac{a}{b}\right)^2+\frac{5}{16}\left(\frac{a}{b}\right)^3\right\}-0{,}47\left(\frac{a}{b}\right)^4+0{,}663\left(\frac{a}{b}\right)^5}{\sqrt{\left(1-\frac{a}{b}\right)^3}}$ $f_2\left(\frac{a}{b}\right) = \left(1-\frac{a}{b}\right)^2 F_1\left(\frac{a}{b}\right)$	LEPŠÍ NEŽ 1% $a < b$

Algorithm of the solution of a body with a crack

1. Determination of the crack size a
2. **Calculation of K-factor**s **from nominal stress**es
3. Comparison of K-factor values for the individual modes
4. Determination of material characteristics (yield stress R_e, fracture toughness K_{IC})
5. **Determination of the plastic zone radius r_k – eq. (11)**
6. **Estimation of LEFM applicability (only if $r_k < a/10$)**
7. **Comparison of the K_I and K_{IC} values.**

Thus the behaviour of the crack under static load is predicted and the factor of safety can be evaluated by using a standard formula:

$$FOS = \frac{K_{IC}}{K_I}$$

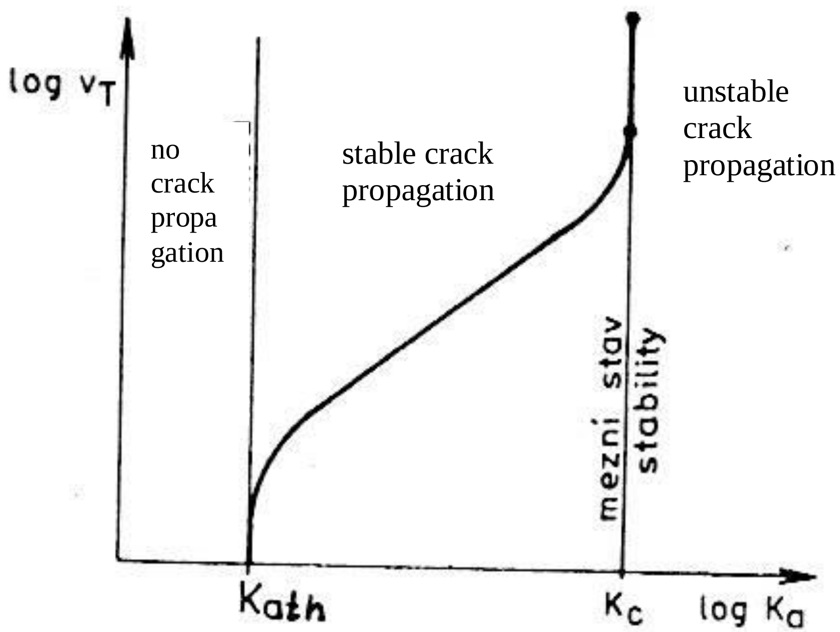
In case of cyclic loading, the following additional steps should be carried out:

8. Calculation of the critical crack length a_c

$$a_c \cong \left(\frac{K_{IC}}{\sigma}\right)^2 \cdot \frac{1}{\pi} \tag{17}$$

This formula is valid approximately for short cracks (less than 1/10 of the cross section) and amplitude K_a of the stress intensity factor between $K_{ath} \le K_a \le K_{IC}$ (stable crack propagation). Its error increases steeply with the crack length.

9. Estimation of the residual lifetime of the body with the crack using Paris-Erdogan formula (18) or directly its integrated form (19).



Stable crack propagation under conditions of cyclic loading (HCF)

The crack growth rate v_T depends on the stress intensity factor K_I. Between its threshold value K_{ath} and the fracture toughness K_{IC} this dependency (see the figure above) can be described by a linear relation (in logarithmic coordinates), which gives (after mathematical removal of logarithms) the following exponential relation (called Paris-Erdogan formula):

$$v_T = \frac{da}{dN} = c\left(\Delta K_I\right)^m \tag{18}$$

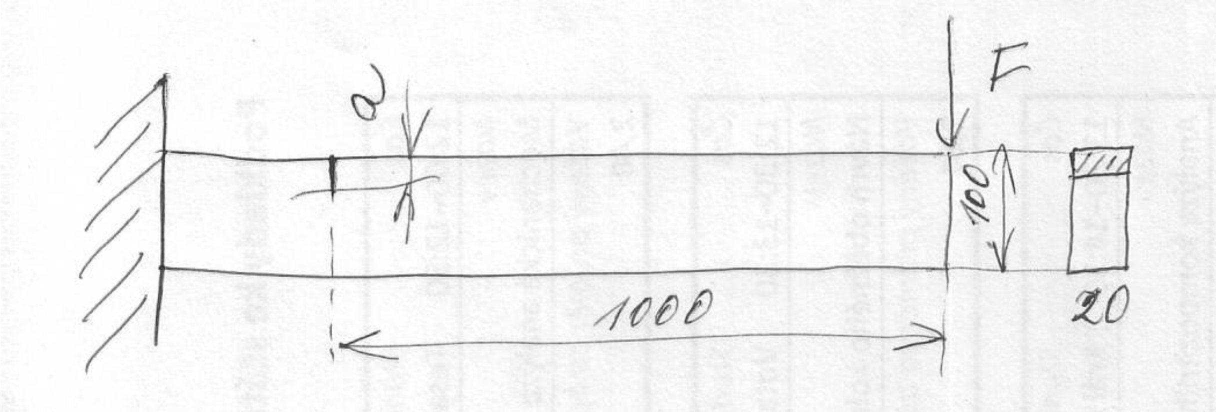
Here the constants *c*,*m* depend on the material and the loading cycle character, respectively. In contrast to the above figure, the range of stress intensity factor $\Delta K_I = 2K_{Ia}$ is used here instead of its amplitude. Through integration of this formula we can obtain the allowable number of the given deterministic cycles with stress range $\Delta\sigma$ in the following form:

$$\Delta N = \frac{2\left(a_f^{1-\frac{m}{2}} - a_0^{1-\frac{m}{2}}\right)}{c\Delta\sigma^m \pi^{\frac{m}{2}}\left(2-m\right)} \tag{19}$$

where ΔN represents the number of cycles before the crack achieves its final (critical or allowable) length a_f when starting from the initial crack length a_0.

Note: It holds for the allowable length $a_f = a_c / FOS$.

Example on LEFM:



Given: K_{IC} = 150 MPa.m$^{-1/2}$; σ_y = 400 MPa; F = 3 kN; a = 5 mm

1. Calculation of nominal stress in the section with the crack

$$\sigma = \frac{6M_o}{bh^2} = \frac{6.3000\,.1000}{20.100^2} = 90\,MPa$$

2. Calculation of K-factor

$$K_I = \sigma\sqrt{\pi a}\left[1{,}112 - 1{,}4\frac{a}{b} + 7{,}33\left(\frac{a}{b}\right)^2 + \dots\right] =$$

$$= 90\sqrt{\pi.0{,}005}\left[1{,}112 - 1{,}4.0{,}05 + 7{,}33(0{,}05)^2 + \dots\right] = 90\sqrt{\pi.0{,}005}.1{,}07 = 12{,}07\,MPa\sqrt{m}$$

3. Determination of the plastic zone radius r_k, with $\alpha=1/3$ for thickness t >10mm (plane strain)

$$r_k = \frac{\alpha}{\pi}\left(\frac{K_I}{\sigma_y}\right)^2 = \frac{1}{3\pi}\left(\frac{12}{400}\right)^2 = 9{,}7.10^{-5}\, m = 0{,}1\, mm$$

4. Estimation of LEFM applicability: it holds $r_k < a/10$.

5. Comparison of the K_I and K_{IC} values:

   $K_I < K_{IC}$ – unstable crack propagation will not occur under the steady load.

For repeated loading cycles:

6. Calculation of the critical crack length a_c

$$a_c \cong \left(\frac{K_{IC}}{\sigma}\right)^2 . \frac{1}{\pi} = \left(\frac{150}{90}\right)^2 . \frac{1}{\pi} = 0{,}88\, m > a$$

Unstable crack propagation cannot occur, because the critical length is higher than the beam thickness.

Estimation of maximum allowable crack length:

$$h_{\min} = \sqrt{\frac{6M_o}{b\sigma_y}} = \sqrt{\frac{18.10^6}{20.400}} = 47\ mm$$

With this thickness plastic deformation occurs without any stress concentration – reduction by a factor of safety is needed. We choose the FOS = 2 and calculate the allowable stress:

$$\sigma_{all} = \frac{\sigma_y}{FOS} = \frac{400}{2} = 200\ MPa$$

Then the minimum height of the loadbearing section is:

$$h_{\min} = \sqrt{\frac{6M_o}{b\sigma_{all}}} = \sqrt{\frac{18.10^6}{20.200}} = 67\ mm$$

and the allowable crack length is

a_{all} = 100-67= 33 mm

Estimation of **residual life** until the crack reaches this length
(m=4 for a repeated cycle, c=7,4.10^{-13} for the given material):

$$\Delta N = \frac{2\left(a_f^{1-\frac{m}{2}} - a_0^{1-\frac{m}{2}}\right)}{c\Delta\sigma^m \pi^{\frac{m}{2}}(2-m)} = \frac{2\left(0,033^{-1} - 0,005^{-1}\right)}{7,4.10^{-13} 90^4 \pi^2 (-2)} = 354000 \quad cycles$$

Residual life is 354 000 cycles but checking the conditions for propagation of the allowable crack is needed:

$$\Delta K_I = K_I = \sigma\sqrt{\pi a}\left[1{,}112 - 1{,}4\frac{a}{b} + 7{,}33\left(\frac{a}{b}\right)^2 + \ldots\right] =$$

$$= 90\sqrt{\pi.0{,}033}\left[1{,}112 - 1{,}4.0{,}33 + 7{,}33(0{,}33)^2 - 13{,}08.(0{,}33)^3 + 14{,}0(0{,}33)^4\right] =$$

$$= 90\sqrt{\pi.0{,}033}.1{,}144 = 33{,}15\,MPa\sqrt{m}$$

$\Delta K_{th} < \Delta K_I < K_{IC}$, it means stable crack propagation as expected.

Plastic zone size:

$$r_k = \frac{\alpha}{\pi}\left(\frac{K_I}{\sigma_y}\right)^2 = \frac{1}{3\pi}\left(\frac{33{,}15}{400}\right)^2 = 7.3.10^{-4}\,m = 0{,}7\,mm < \frac{33\,mm}{10}$$