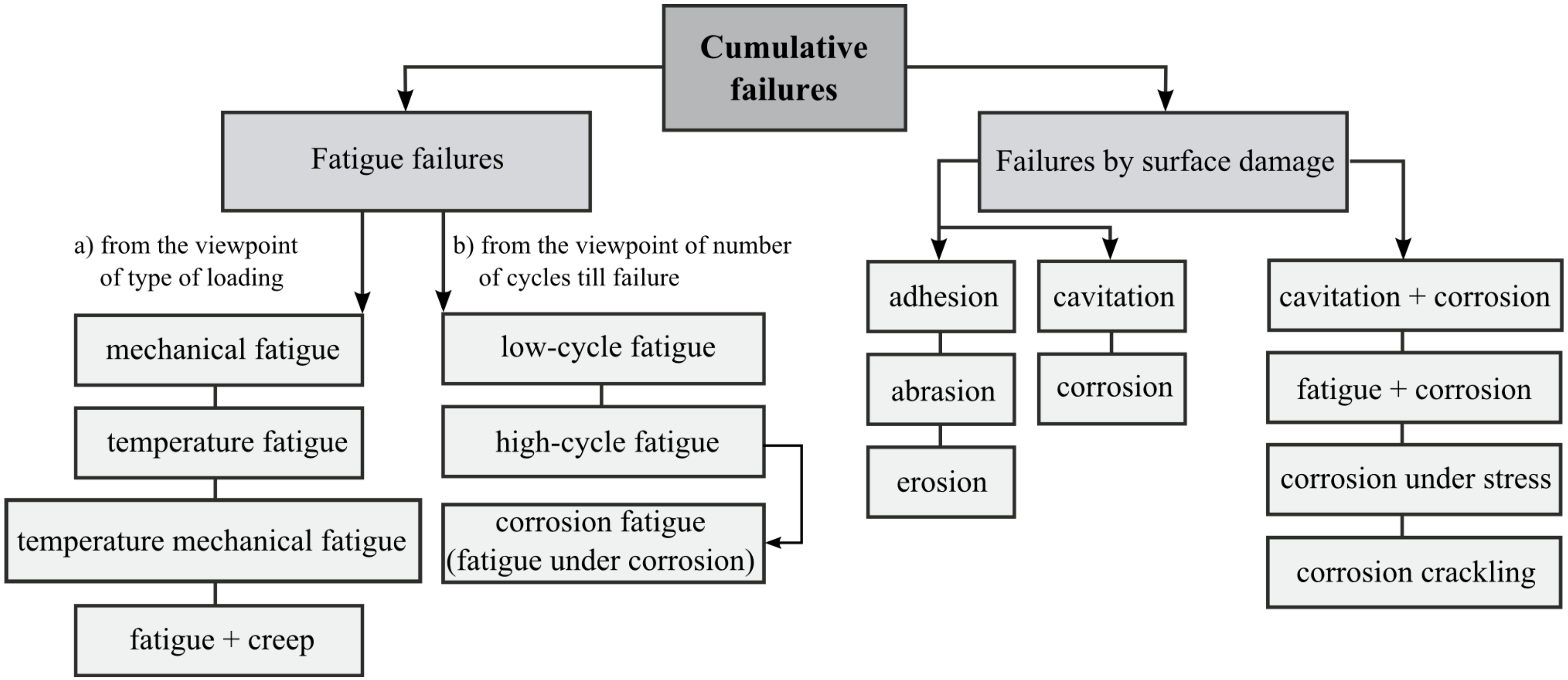
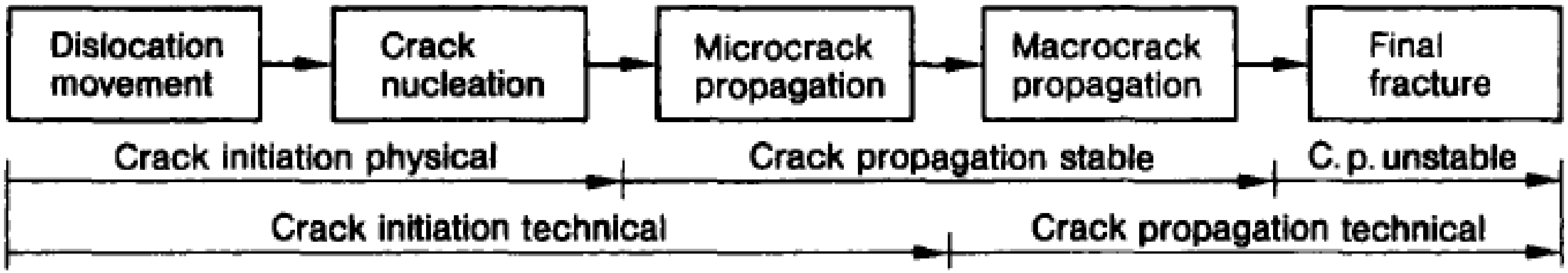
Fatigue failure

is one of the so called cumulative limit states. In opposite to instantaneous limit states, the cumulative ones depend not only on the instantaneous loading (stress-strain) state of the body in question but on all its loading history. During the body operation and loading, irreversible changes in the material occur, as well as accumulation of damage of the body. Outer influencing factors, like temperature, surface quality, chemical effect of the surrounding medium, energy fields, etc. play a significant role in origination of the cumulative limit states.

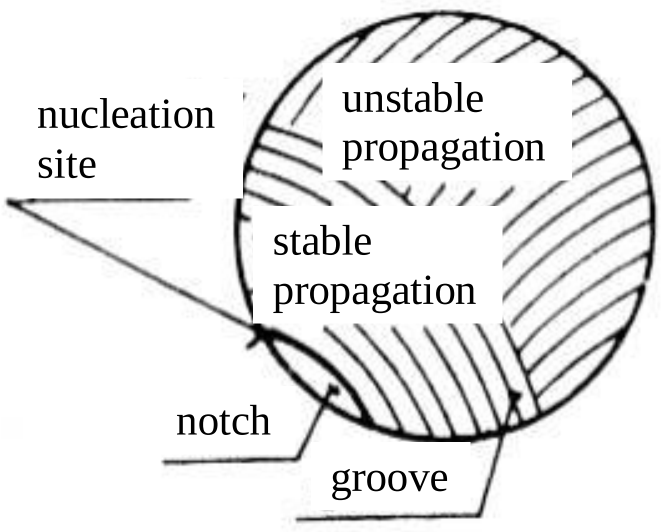
Cumulative failures (limit states) can be systemized as follows:



Stages of the fatigue process from the microscopic and macroscopic point of view



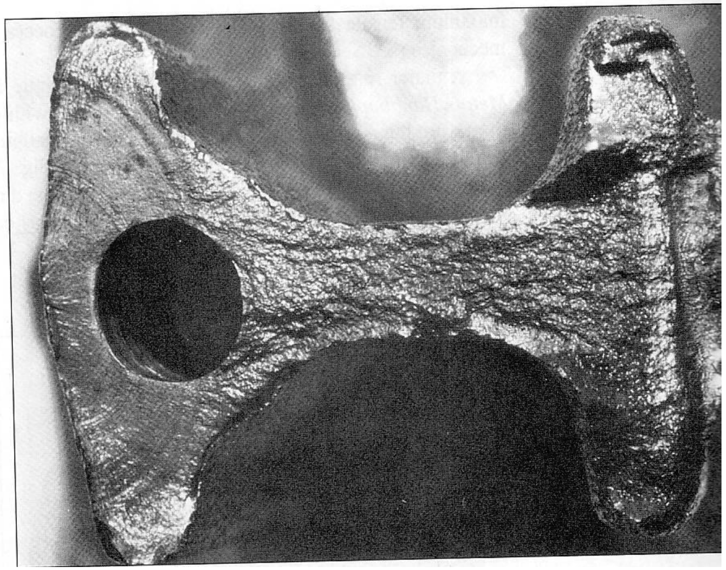
Classical approaches (Wöhler) do not deal with the initiation, existence or propagation of a crack but only with fatigue failure (fracture) of the body, which causes termination of the body lifetime. Behaviour of an existing crack and prediction of its propagation is described by methods of ***fracture mechanics***.

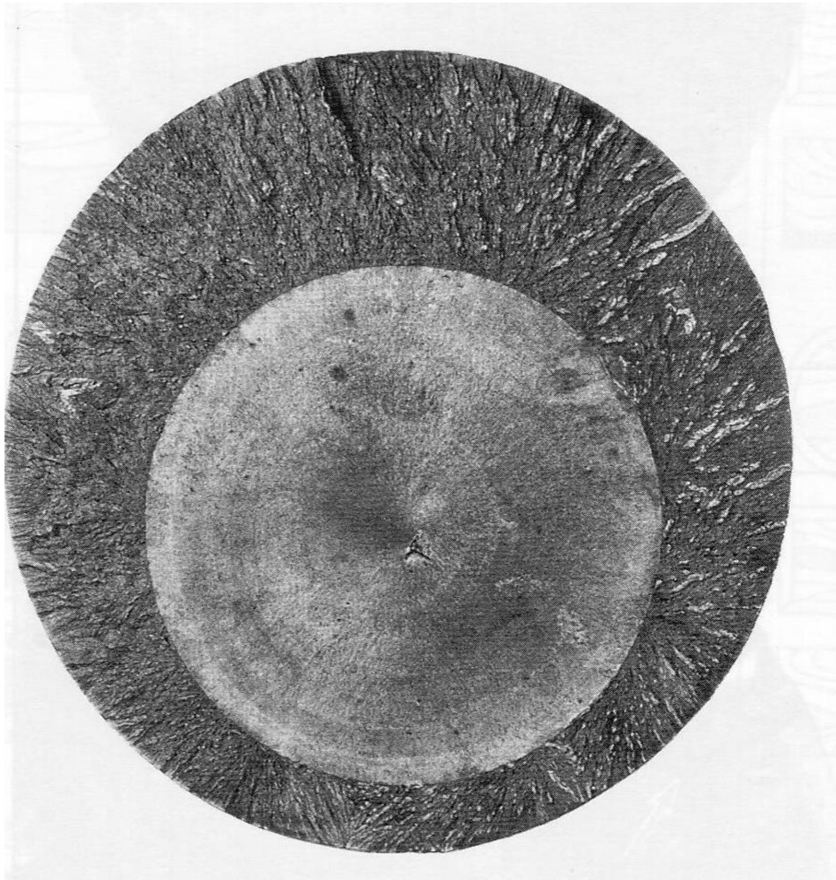


Examples of fatigue fractures under tension

Figure 7–5

Fatigue fracture surface of a forged connecting rod of AISI 8640 steel. The fatigue crack origin is at the left edge, at the flash line of the forging, but no unusual roughness of the flash trim was indicated. The fatigue crack progressed halfway around the oil hole at the left, indicated by the beach marks, before final fast fracture occurred. Note the pronounced shear lip in the final fracture at the right edge. *(From ASM Handbook, Vol. 12: Fractography, ASM International, Materials Park, OH 44073-0002, fig 523, p. 332. Reprinted by permission of ASM International®, www.asminternational.org.)*



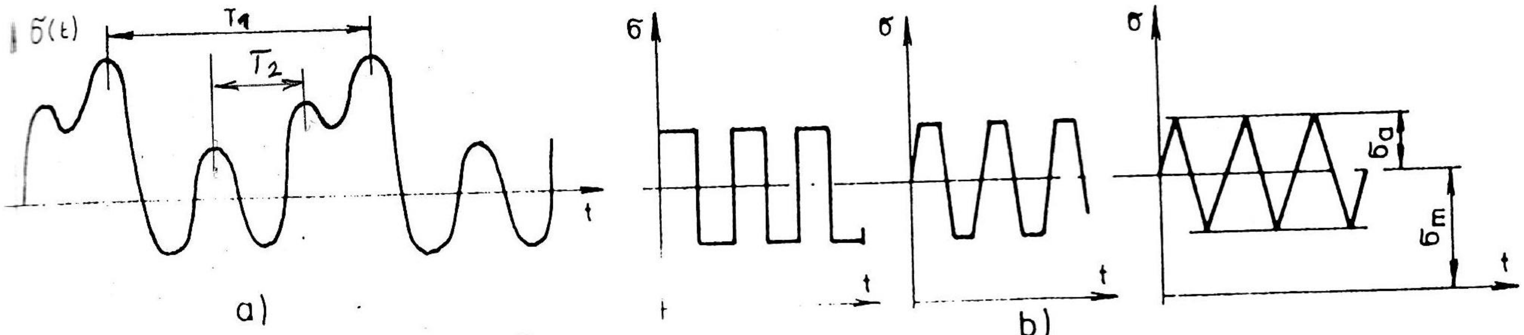


Fatigue fracture surface of a 200-mm (8-in) diameter piston rod of an alloy steel steam hammer used for forging. This is an example of a fatigue fracture caused by pure tension where surface stress concentrations are absent and a crack may initiate anywhere in the cross section. In this instance, the initial crack formed at a forging flake slightly below center, grew outward symmetrically, and ultimately produced a brittle fracture without warning. *(From ASM Handbook, Vol. 12: Fractography, ASM International, Materials Park, OH 44073-0002, fig 570, p. 342. Reprinted by permission of ASM International®, www.asminternational.org.)*

Fatigue failure occurs under variable stresses and strains, existing mostly under loads varying in time (but it can occur sometimes under steady or monotonically increasing loads – vibrations caused by velocity of the surrounding medium or bending of rotating beams). It is hereditary (depends on the loading history) in consequence of accumulation of damage caused by individual loading cycles.

The stresses and strains varying in time can be:

- deterministic
  - periodic
    - harmonic
    - non-harmonic
  - non-periodic
- stochastic
  - stationary
  - non-stationary



Obr. 154

Shape and frequency of the cycles do not mostly influence the fatigue damage significantly. In computational evaluation of fatigue, the order of loading cycles is often not taken into account as well.

Basic parameters of a stress cycle:

Mean (midrange) stress: σ_m

Stress amplitude: σ_a

Stress range: $\Delta\sigma = 2\sigma_a$

Minimum stress: $\sigma_n = \sigma_{min} = \sigma_m - \sigma_a$

Maximum stress: $\sigma_h = \sigma_{max} = \sigma_m + \sigma_a$

Stress ratio

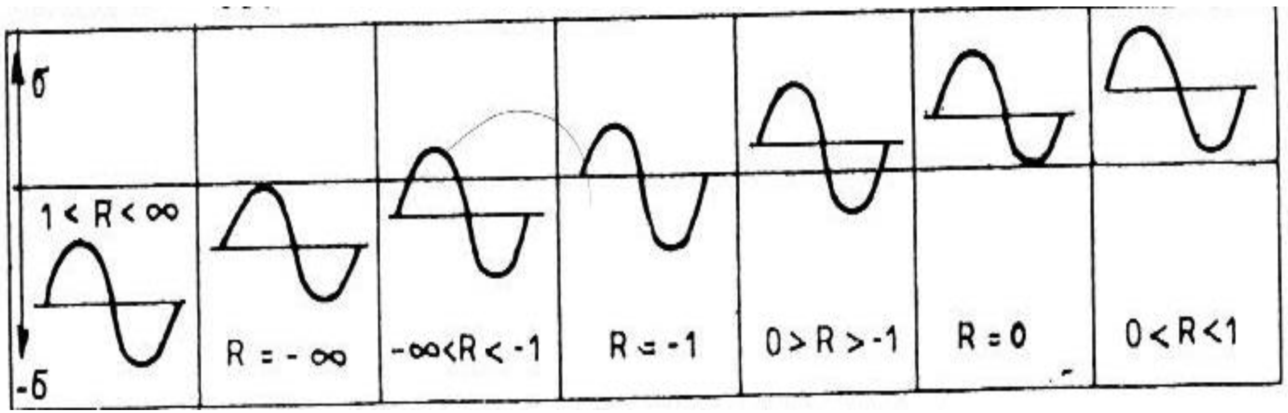
(coefficient of asymmetry): $R = \sigma_n / \sigma_h = \sigma_{min} / \sigma_{max}$

Period of cycle: T[s]

Frequency: f = 1/T

Basic types of cycles and their asymmetry coefficients:

1. Fluctuating in compression
2. Repeated (pulsating) in compression
3. Asymmetrical reversed
4. Completely reversed (symmetric)
5. Asymmetrical reversed
6. Repeated (pulsating) in tension
7. Fluctuating in tension



Fatigue characteristics of structural members do not depend only on material (here much more on its crystalline structure) but also on their:

- shape including stress concentrations (notches),
- size and non-homogeneity of stress state,
- heat and mechanical treatment,
- surface quality and surface treatment,
- surrounding conditions (temperature, corrosion accelerated by salt water...).

Consequently there are two types of fatigue characteristics:

- component specific,
- material specific – **basic fatigue characteristics.**

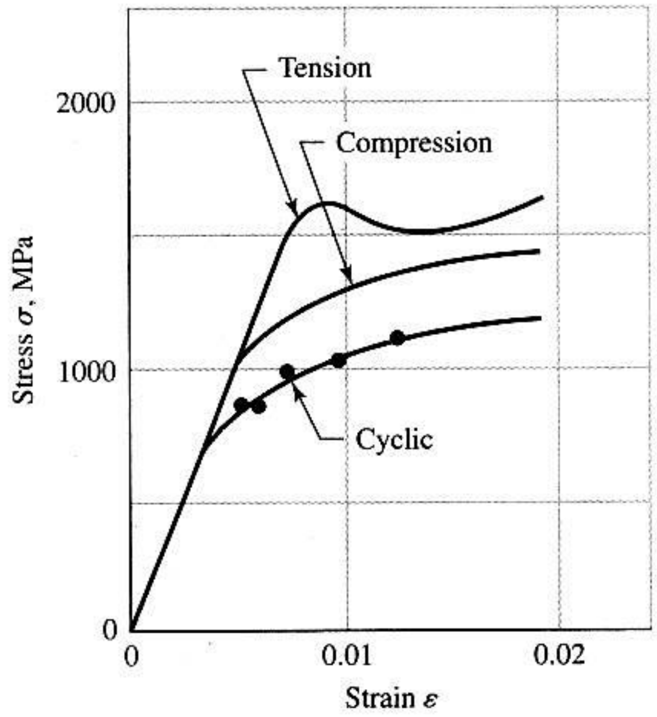
Basic fatigue characteristics:

- **Cyclic stress-strain curve** σ – ε , described by Ramberg-Osgood approximative relation
  $\sigma_a = K'.\varepsilon_{ap}^{n'}$,

  $$\varepsilon_a = \varepsilon_{ae} + \varepsilon_{ap} = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'}\right)^{\frac{1}{n'}} \qquad (1)$$

  can show cyclic strain hardening or softening, i.e. stiffening or softening against the static curve σ-ε.

  Cyclic strain hardening is typical for low strength carbon steels with $\frac{R_m}{R_e} = \frac{\sigma_u}{\sigma_y} > 1{,}4$

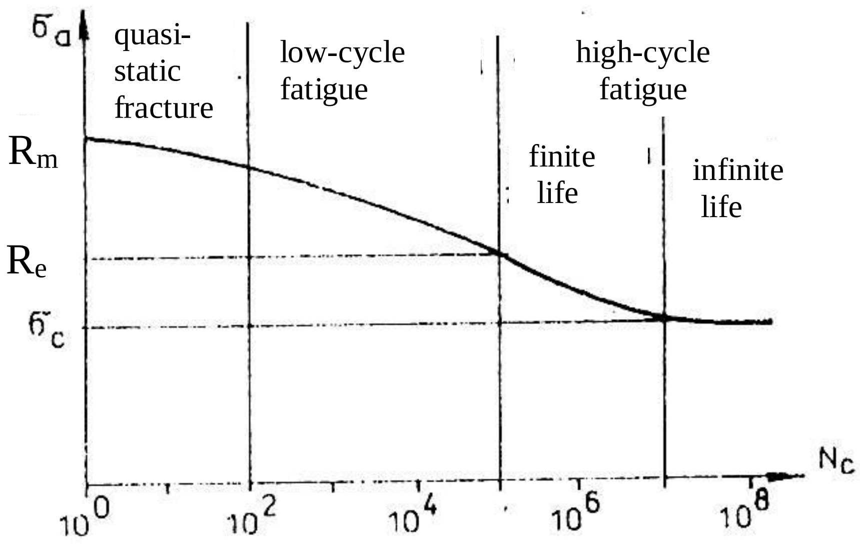


Cyclic strain softening is typical for high strength steels and alloys with $\frac{R_m}{R_e} = \frac{\sigma_u}{\sigma_y} < 1{,}2$

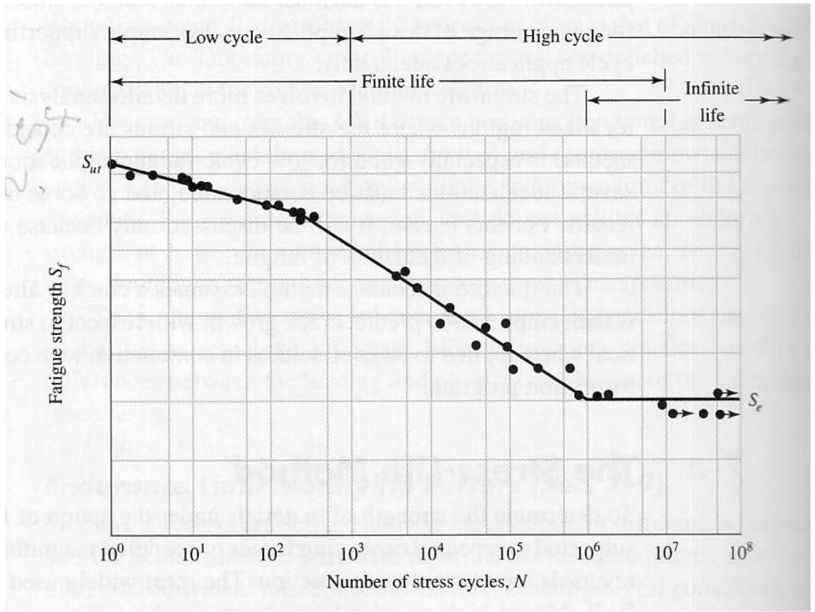
- **S-N curve (Wöhler** curve) is dependence of number of cycles until fracture[1] on the stress amplitude of a symmetric loading cycle (σ_m=0) – applicable for high-cycle fatigue (HCF), it defines also the endurance limit σ_C if it exists (it is not the case e.g. for aluminium); the fatigue strength (for finite life under high cycle fatigue) can be described e.g. by the equation:

$$\log N_f = a - m.\log \sigma_a \Rightarrow \sigma_a^m.N = \sigma_C^m.N_C = A \qquad (2)$$

According EU conventions



According US conventions

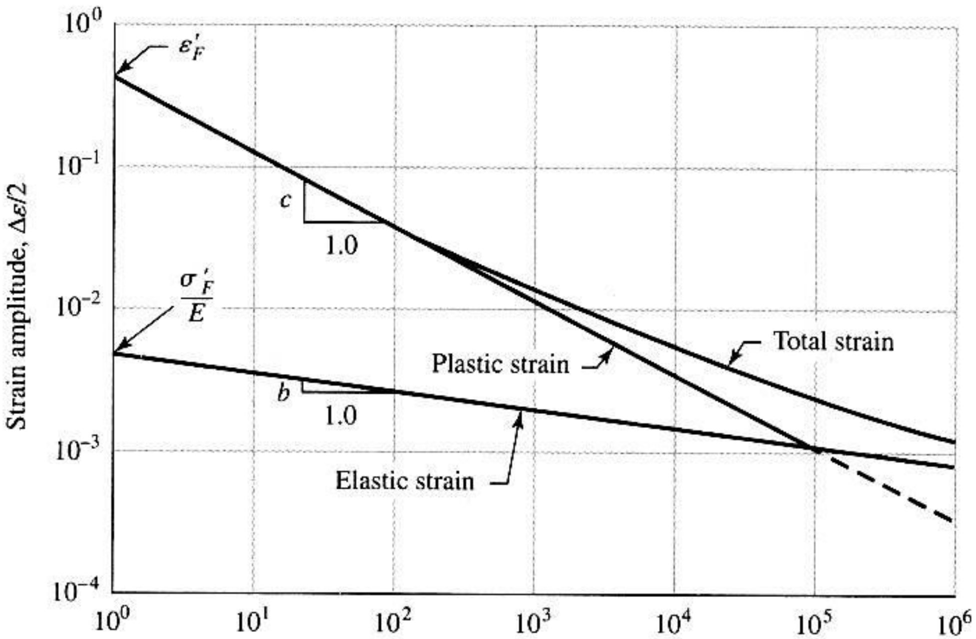


Specific parameters of the curve are valid always for a certain probability of failure and confidence.

---

[1] The definition of fatigue failure is conventional, alternatively to fatigue fracture it can be given also by initiation of a crack of a defined size or by a defined decrease of stiffness of the specimen during the fatigue test, as consequence of crack propagation.

- **Manson-Coffin curve**[2] – applicable for low-cycle fatigue (LCF), depicted in logarithmic coordinates ($\log\varepsilon_a$-$\log N$); its mathematical description is based on the elastic and plastic components of strain amplitude



Number of reversals to failure, $2N_f$

$$\varepsilon_{at} = \varepsilon_{a,e} + \varepsilon_{a,p} = \frac{\sigma'_f}{E}\left(2N_f\right)^b + \varepsilon'_f\left(2N_f\right)^c \tag{3}$$

In the formula of Manson-Coffin curve, the meaning of symbols is as follows:

N_f – number of cycles to failure
ε'_f – fatigue ductility coefficient
σ'_f – fatigue strength coefficient
c – fatigue ductility exponent
b – fatigue strength exponent

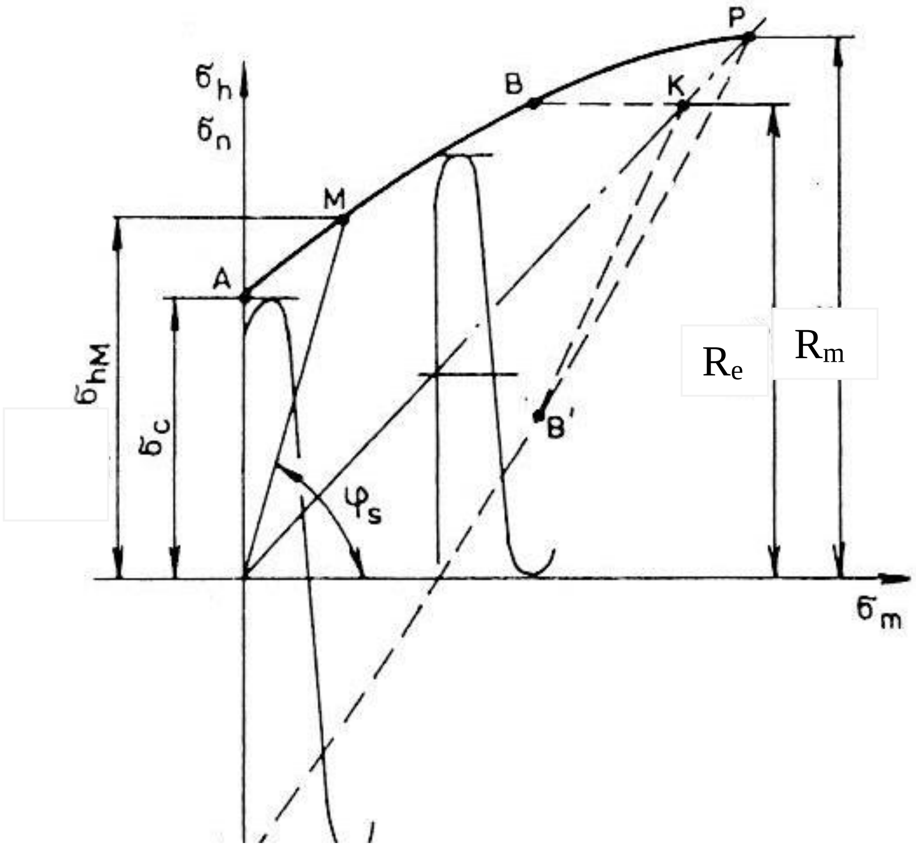
The boundary between HCF and LCF is conventional, from 10^5 (in EU countries) down to 10^3 (in USA) cycles till failure.

[2] The dependency of total strain ε_{at} on the number of cycles till failure N_f is mostly called Manson-Coffin curve in literature, although these authors have proposed this dependence for the plastic strain component only and it was Basquin and Morrow who extended it into the form presented here.

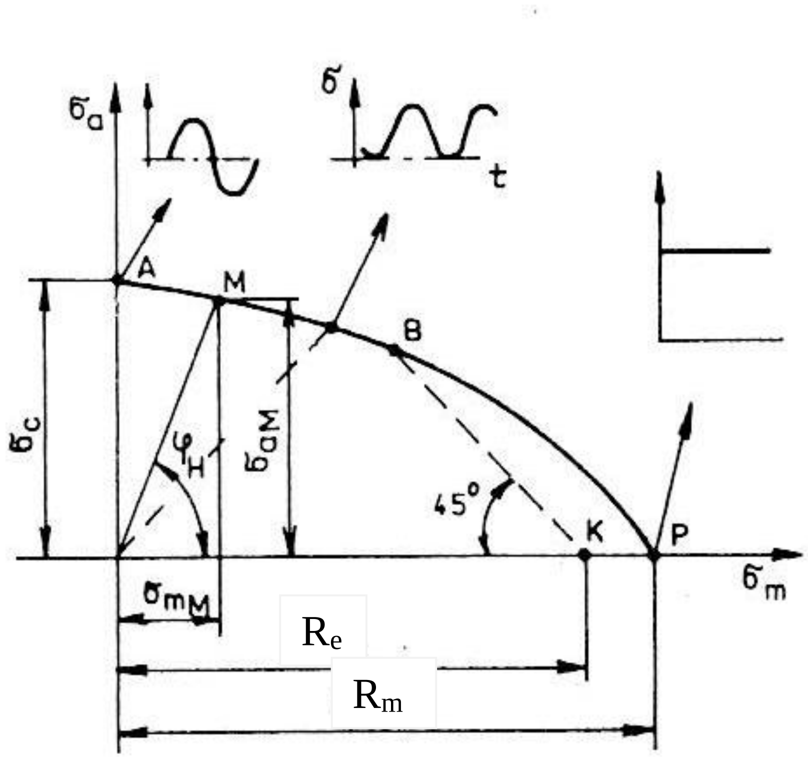
Basic fatigue characteristics are determined for uniaxial stress state (in tension or bending, eventually in torsion – shear stress state) and **for symmetric (completely reversed) cycles** ($\sigma_m = 0$, or R = -1).

For asymmetric cycles ($\sigma_m > 0$) the endurance limit (or fatigue strength) is determined from **Smith** or **Haigh** diagrams, representing additional fatigue characteristics.

Smith diagram

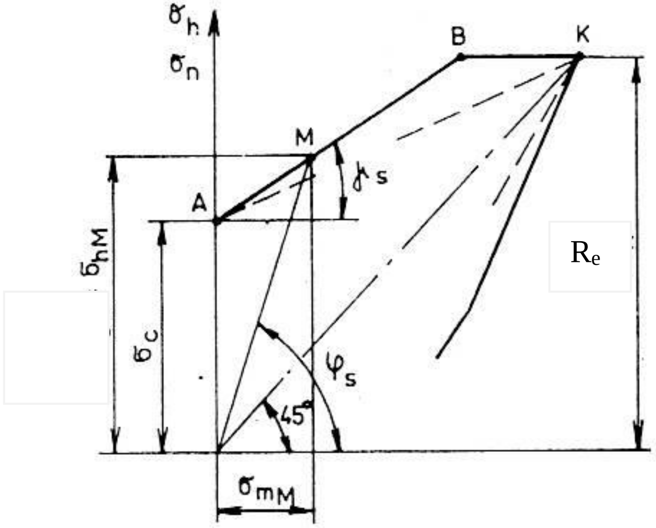


Haigh diagram

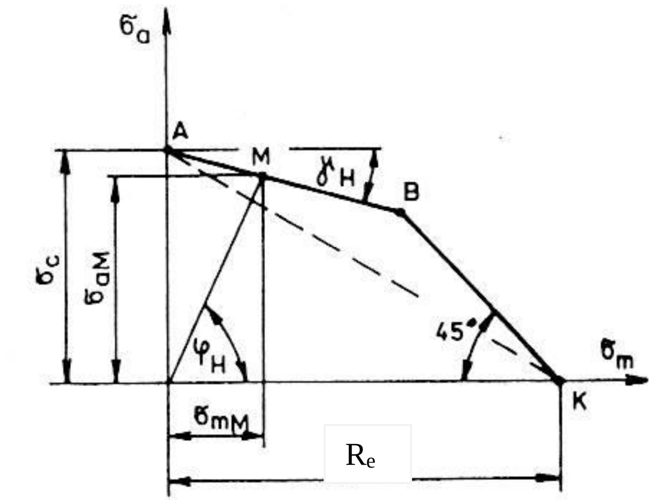


Simplified diagrams (Serensen approach with slope of limit lines as function of strength of material)

Smith diagram



Haigh diagram



Angles γ_S and γ_H used in this simplified diagrams can be calculated from the following relations:

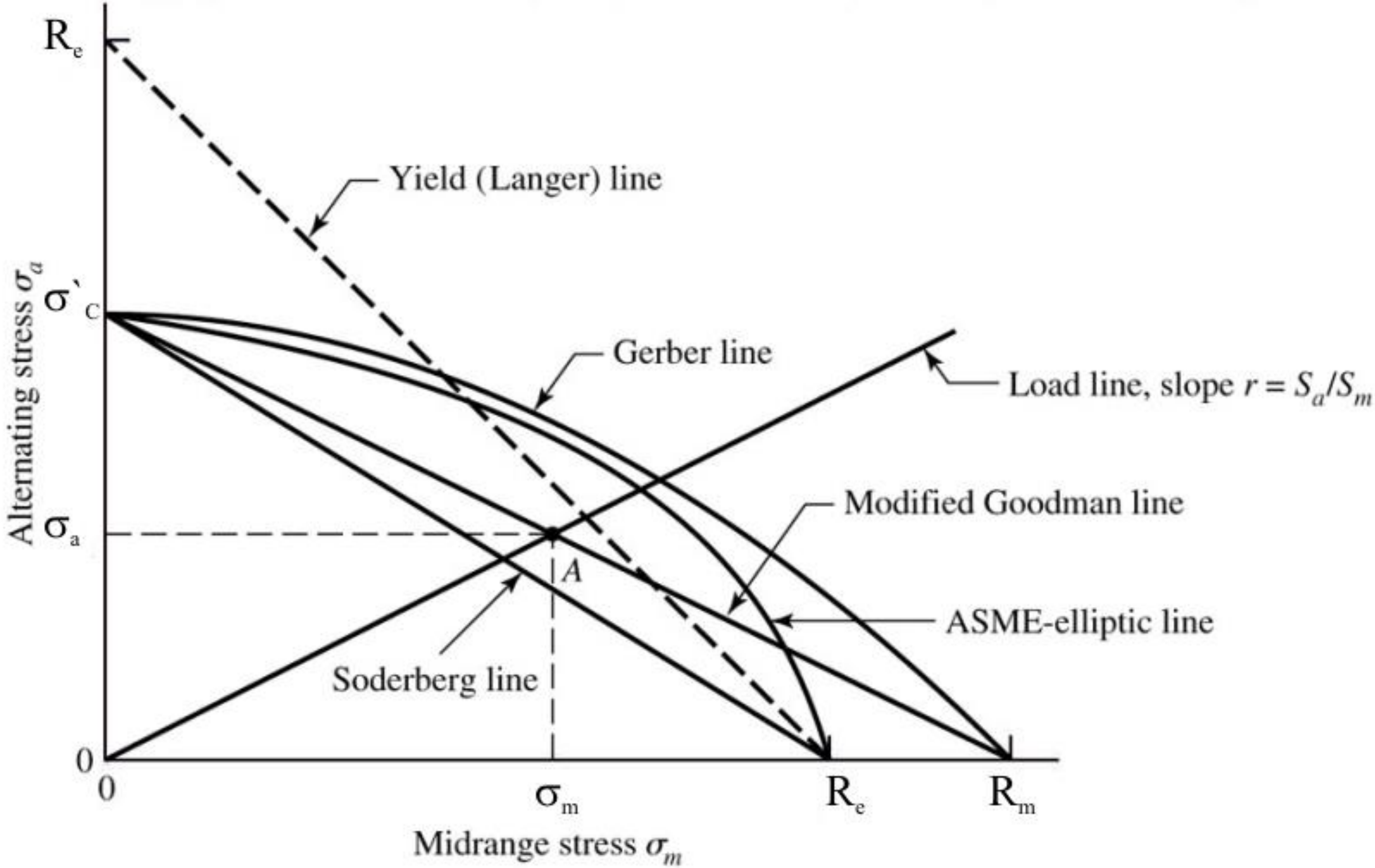
$$tg\gamma_S = 1 - \psi \qquad tg\gamma_H = \psi = \frac{\sigma_c - \sigma_{aM}}{\sigma_{mM}} \qquad (4)$$

where constants ψ can be taken from the table below:

R_m [MPa]	350-520	520-700	700-1000	1000-1200	1200-1400
ψ_σ	0	0,05	0,1	0,2	0,25
ψ_τ	0	0	0,05	0,1	0,15

Note: *Smith and Haigh diagrams, as well as their simplified forms, can be created also for shear stresses; subscripts σ and τ relate here to normal and shear components of stress, respectively.*

Other simplifications of Haigh diagram



Mathematical description of boundary lines (limit envelopes)

related to limit envelopes for infinite life and concept of local stresses

(can be also formulated similarly for finite life and concept of nominal stresses).

Soderberg (linear) –suitable for body without notches, otherwise very conservative, because it excludes plastic deformations totally.

$$\frac{\sigma_a}{\sigma'_C}+\frac{\sigma_m}{R_e}=1\rightarrow \sigma_a=\sigma'_C-\frac{\sigma'_C}{R_e}\sigma_m$$

The other criteria should be applied in combination with Langer line to avoid plastic deformations (LCF).

Goodman (linear)

$$\frac{\sigma_a}{\sigma'_C}+\frac{\sigma_m}{R_m}=1\rightarrow \sigma_a=\sigma'_C-\frac{\sigma'_C}{R_m}\sigma_m$$

Gerber (parabolic)

$$\frac{\sigma_a}{\sigma'_C}+\left(\frac{\sigma_m}{R_m}\right)^2=1\rightarrow \sigma_a=\sigma'_C-\frac{\sigma'_C}{R_m^2}\sigma_m^2$$

ASME (eliptic)

$$\left(\frac{\sigma_a}{\sigma'_C}\right)^2+\left(\frac{\sigma_m}{R_e}\right)^2=1$$

Serensen[3]

$$\sigma_a=\sigma'_C-\psi_\sigma\frac{\sigma'_C}{\sigma_C}\sigma_m$$

or the simplified formula

$$\sigma_a=\sigma'_C-\psi_\sigma\sigma_m$$

Accuracy and consequently applicability of the individual criteria depends on the type of material and can be assessed only by their statistical comparison with experimental results.

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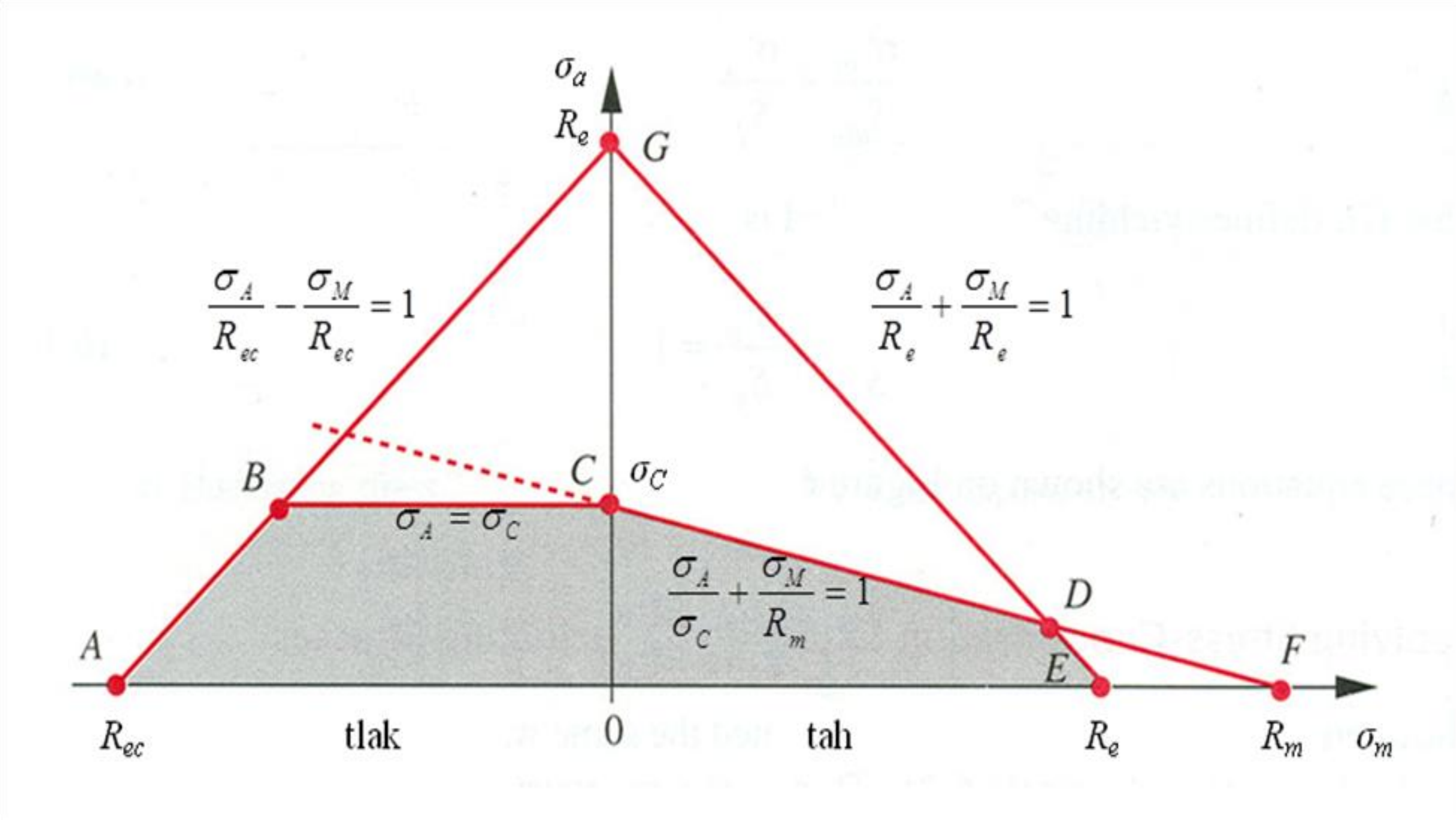
[3] Approach presented in the Czech textbook Ondráček, Vrbka, Janíček, Burša: Pružnost a pevnost II.

Simplified **Haigh diagram** (Goodman approximation) **extended for compressive region** ($\sigma_m < 0$)



Note: This simplification does not take negative (compressive) mean stress into consideration until the yield stress is reached. In fact, also negative mean stresses can influence (increase, in opposite to tensile mean stresses, see the dashed line) the endurance limit; the approaches taking this into account are, however, out of scope of this course.

Computational assessment of fatigue failure

Overview of concepts for assessment of non-welded structures

1. **Concept of nominal stresses**
   - one-stage loading (constant stress amplitude as well as midrange stress)
     - infinite life
       - uniaxial stress state
       - biaxial stress state
     - finite life
   - multi-stage deterministic loading (several various types of stress cycles)
   - random (stochastic) loading (variable stress amplitude) under uniaxial stress state

2. **Concepts of local stresses and strains**
   - concept of local elastic stresses
   - concepts of local elastic-plastic stresses and strains – for finite life in the LCF region
     - Neuber concept
     - concept of equivalent energy (Molski – Glinka)
     - many other concepts

3. **Concepts of fracture mechanics**
   - concepts describing a stable crack propagation

Concept of nominal stresses – for infinite life

The procedure described here holds for a simple case – **uniaxial stress state, infinite life** and completely reversed (symmetric) cycle. For asymmetric cycle Haigh diagram is to be applied.

Criterion of fatigue failure is $$\sigma_{a,nom} = \sigma_C^*,$$

and consequently the factor of safety $$k_C = \frac{\sigma_C^*}{\sigma_{a,nom}}$$

or $$k_C = \frac{\sigma_{hC}^*}{\sigma_{h,nom}}$$ for a repeated cycle, where

$\sigma_{\mathrm{a,nom}}$ – amplitude of nominal stress,

σ_C^* – endurance limit of a notched part in a completely reversed cycle,

$\sigma_{h,nom}$ – maximum stress of the cycle,

σ_{hC}^* – endurance limit of a notched part (maximum stress) in a repeated cycle.

Both formulas for the factor of safety (calculated on the basis of either amplitudes or maximum stresses of the cycle) are **equivalent in case of proportional loading** and overloading process. For non-proportional loading process (mean stress is not proportional to the stress amplitude) they yield different results and consequently also different values of the factor of safety are recommended for both approaches.

In our course the formula related to **stress amplitude is preferred**.

Determination of endurance limit

Endurance limit can be determined either experimentally using the produced component part directly, or by recalculation from the basic endurance limit σ_c using e.g. relations with respective corrections factors:

$$\sigma_C^* = \sigma_C \frac{\nu\eta}{\beta} \quad , \qquad \tau_C^* = \sigma_C \frac{\nu\eta}{\beta} \tag{5}$$

In this concept the endurance limit is lowered by the notch factor β against a smooth specimen, so that it corresponds to the endurance limit determined experimentally with the real component part. The symbols:

σ_C – basic endurance limit (for a smooth specimen without notch in tension-compression symmetric cycle)
ν – size factor
η – surface condition factor
β – notch factor ($\beta > 1$)

Size factor $\nu = \nu_1 . \nu_2$, thus it consists of two parts:

ν_1 – size factor itself, decreases with increasing size of the evaluated body,

ν_2 – factor representing non-homogeneity of stress in bending or torsion (increases with stress gradient in the body which is also size-dependent).

Surface condition factor consists also of two parts $\eta = \eta_1 . \eta_2$; here $\eta_1 < 1$ represents impact of surface roughness and corrosive surrounding medium, while $\eta_2 > 1$ is the impact of heat, chemical or mechanical surface treatment (surface hardening, nitridation, cementation, etc.).
These factors can be assessed on the basis of empirical formulas or graphs presented in literature (e.g. Ondráček et al., pages 211-214).

Notch factor *β* represents a portion of the shape factor (stress concentration factor) α, which applies under repeated loading; it meets the limits 1 < β < α and can be determined in different ways:

- using Heywood formula

$$\beta = \frac{\alpha}{1+2\frac{(\alpha-1)}{\alpha}\sqrt{\frac{a'}{r}}} \quad \text{or} \quad \beta = \frac{\alpha}{1+\frac{(\alpha-1)}{\alpha}\frac{K}{\sqrt{r}}} \qquad \text{(6a)} \quad \text{(6b)}$$

where *r* is the notch radius [mm] and K for eq. (6b) can be found in Ondráček et al. at page 213.

- According Neuber from the formula $\beta = 1+\frac{\alpha-1}{1+\sqrt{\frac{a}{r}}}$

In both approaches the parameter *a* (or *a’*) is determined on the basis of experiments.

- Also the notch sensitivity factor *q* can be introduced by the following formula: $q = \frac{\beta-1}{\alpha-1}$

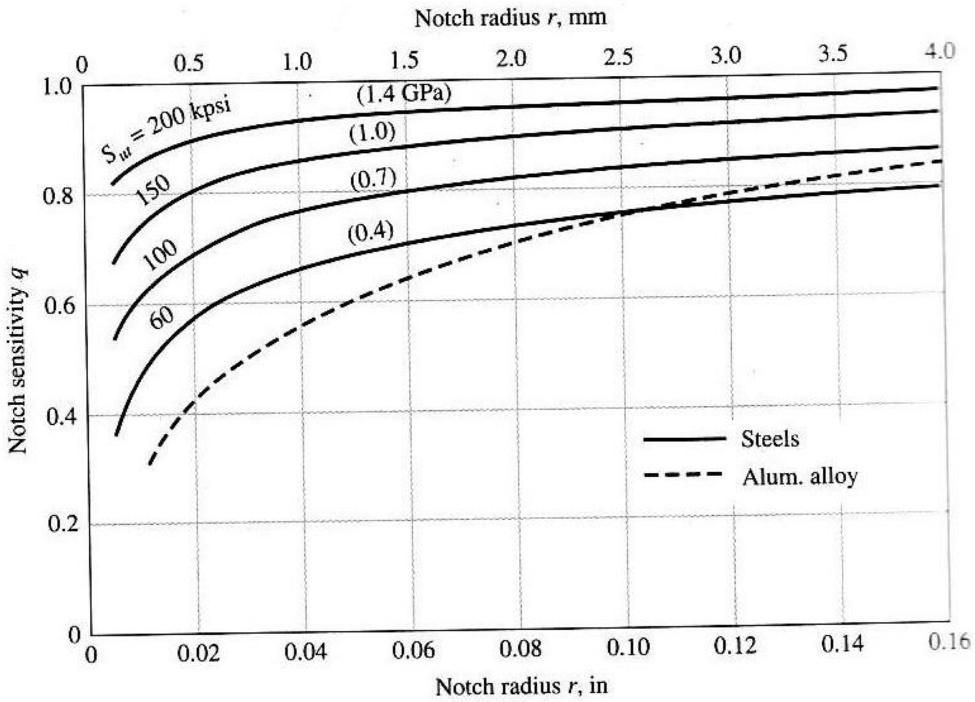
from which it holds $\beta = 1+q(\alpha-1)$

If the notch sensitivity factor q is expressed from Neuber formula, we obtain (Shigley,eq. 7-33)

$$q = \frac{\beta - 1}{\alpha - 1} = \frac{1}{1 + \frac{\sqrt{a}}{\sqrt{r}}}$$

For some frequently applied materials the notch sensitivity was expressed graphically (see Shigley, fig. 7-20)

Always it holds **1< β < α**



For steels the graph can be approximated by Neuber equation where a is substituted by (Shigley, eq. 7-34)

$$\sqrt{a} = 1,238 - 0,225 \cdot 10^{-2} \cdot R_{\mathrm{m}} + 0,160 \cdot 10^{-5} \cdot R_{\mathrm{m}}^{2} + 0,410 \cdot 10^{-9} \cdot R_{\mathrm{m}}^{3}$$

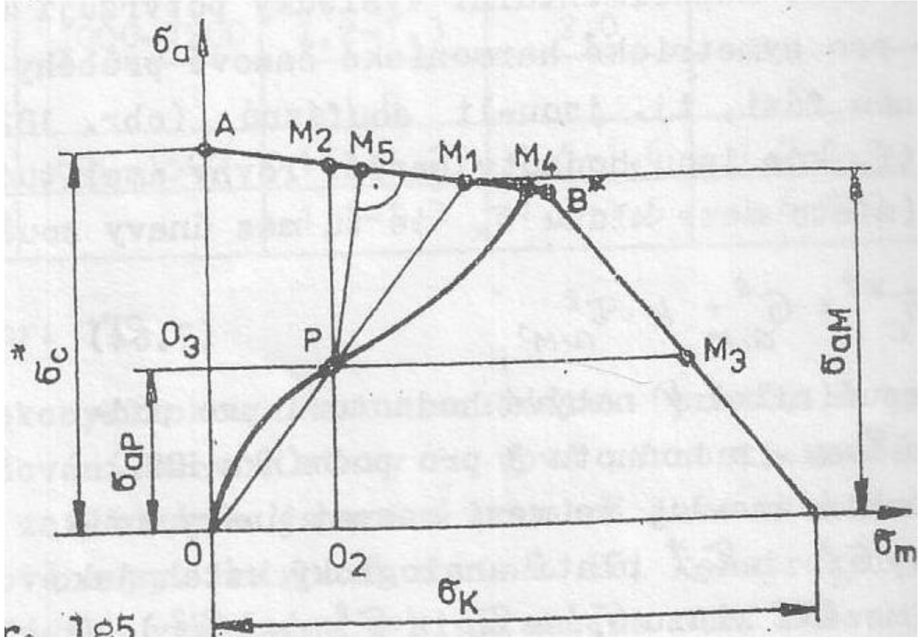
For **asymmetric cycles** (reversed, repeated, or fluctuating), Haigh diagram is to be applied. In most cases it is created for a specific component directly; the choice of its simplification depends chiefly on availability of experimental data for the given material[4].

Note: Haigh diagram can be used for component parts in both concepts of nominal stresses and of local elastic stresses. The most important difference is introduction of the notch factor either in calculation of stresses (concept of local elastic stresses) or in reduction of the endurance limit (concept of nominal stresses). This difference is distinguished by using different symbols (σ'_C or σ^*_C) for both quantities used alternatively in creation of the limit envelope in Haigh diagram. In assessment of infinite life, the following quantities can then be used as limit values:

- σ^*_C … endurance limit of a notched part. This represents nominal stress (i.e. stress calculated using elementary formulas of simple theory of elasticity), it applies **in the concept of nominal stresses**.

- σ'_C … limit notch stress. This represents a limit stress value in the root of the notch in the part, under which fatigue failure occurs. It applies **in the concept of local elastic stresses** as presented later.

[4] The diagrams are usually drawn for infinite life (as shown earlier) but they can be applied also for finite life in the HCF region.



For a **non-proportional loading** process represented by general (non-linear) loading and overloading trajectories in the figure, the **factor of safety** is defined as a **ratio of lengths** of the overloading trajectory (till failure represented by different points M_i in the figure) to the loading trajectory (till the operation point P).

Under assumption of a **proportional loading and overloading process** (i.e. ratio of the amplitude and mean stress remains constant during the process, see the linear trajectory OPM_1 in the figure), the above ratio of trajectories can be recalculated into the following formulas defining the **factor of safety** $\boldsymbol{k_c}$, depending on the applied approximation of the limit envelope. Using **Serensen** approximation we obtain:

$$k_C = \frac{\sigma_C^*}{\dfrac{\sigma_C^*}{\sigma_C}\psi_\sigma \sigma_m + \sigma_a} \cong \frac{\sigma_C^*}{\sigma_{ae}} \qquad \text{or} \qquad k_C = \frac{\tau_C^*}{\dfrac{\tau_C^*}{\tau_C}\psi_\tau \tau_m + \tau_a} \cong \frac{\tau_C^*}{\tau_{ae}} \tag{7}$$

where σ_{ae} (τ_{ae}) represents amplitude of an equivalent completely reversed cycle (i.e. for $\sigma_m = 0$, $\tau_m = 0$) with the same factor of safety; if ($\psi_\sigma . \sigma_m$) is much lower than σ_a or the ratio σ_C^* / σ_C is close to 1, this equivalent stress amplitude can be assessed also using the following simplified formulas:

$$\sigma_{ae} \cong \psi_\sigma \sigma_m + \sigma_a \qquad \text{or} \qquad \tau_{ae} \cong \psi_\tau \tau_m + \tau_a \tag{8}$$

For the **concept of local elastic stresses,** a similar approach is applicable only for a body **without notches**; depending on the applied simplification of Haigh diagram the following formulas for the amplitude σ_{ae} of an equivalent completely reversed cycle can be obtained (for a proportional loading and overloading process):

Soderberg:

$$\frac{\sigma_a}{\sigma'_C}+\frac{\sigma_m}{R_e}=\frac{1}{k}\rightarrow k=\frac{\sigma'_C}{\sigma_a+\dfrac{\sigma'_C\,\sigma_m}{R_e}} \qquad \sigma_{ae}=\frac{\sigma'_C}{R_e}\sigma_m+\sigma_a$$

Goodman:

$$\frac{\sigma_a}{\sigma'_C}+\frac{\sigma_m}{R_m}=\frac{1}{k}\rightarrow k=\frac{\sigma'_C}{\sigma_a+\dfrac{\sigma'_C\,\sigma_m}{R_m}} \qquad \sigma_{ae}=\frac{\sigma'_C}{R_m}\sigma_m+\sigma_a$$

The non-linear approximations of Haigh diagram **disable application of the equivalent stress amplitude**.

For **ASME criterion** the factor of safety of a non-symmetric cycle can be expressed as follows:

$$\left(\frac{k\sigma_a}{\sigma'_C}\right)^2+\left(\frac{k\sigma_m}{R_e}\right)^2=1\rightarrow k=\sqrt{\frac{1}{\left(\dfrac{\sigma_a}{\sigma'_C}\right)^2+\left(\dfrac{\sigma_m}{R_e}\right)^2}}$$

For **Gerber** (parabolic) **criterion** a simple explicit expression of the factor of safety is not more possible; the factor of safety *k* can be calculated by solving the following quadratic equation:

$$\frac{k\sigma_a}{\sigma'_C}+\left(\frac{k\sigma_m}{R_m}\right)^2=1$$

In case of **non-proportional loading process** (mean stress is not proportional to the stress amplitude, loading trajectory is curvilinear in Haigh diagram) the factor of safety cannot be calculated from the above formulas, it should be evaluated on the basis of loading and overloading trajectories in Haigh diagram:

$$k_C=\frac{OM}{OP}\,, \tag{9}$$

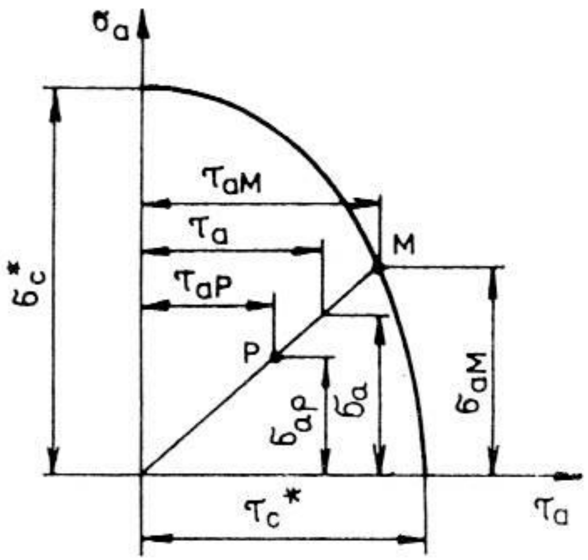
where OM – length of loading and overloading trajectory from the origin to the limit point M,
OP – length of loading trajectory from the origin to the operation point P.

For some of the loading and overloading trajectories the resulting values of the factor of safety can be substantially different from those valid for the proportional loading process.

If Smith diagram is applied to evaluate these non-proportional factors of safety, the resulting values differ from those evaluated on the basis of Haigh diagram. Therefore different values of factors of safety are recommended for them in practical applications.

Safety under combined loading (biaxial state of stress)

Under combined loading of bars (beams or shafts, with both normal and shear components of stresses being non-zero), calculation of the factor of safety is based on a graphical representation of the limit envelope in coordinates presented in the figure. The approach is based on assumption of the same frequency and phase of all the evaluated (completely reversed – symmetric) stress cycles. If it is not the case, the calculated factor of safety is more conservative (lower than in reality).



For **proportional loading process** the factor of safety can be calculated separately for normal ($k_{c\sigma}$) and shear ($k_{c\tau}$) stresses; on the basis of the graphical representation (equation (11) of the ellipse) the formula can be derived for the resulting factor of safety:

$$k_C = \frac{k_{c\sigma} \,.\, k_{c\tau}}{\sqrt{k_{c\sigma}^2 + k_{c\tau}^2}} \qquad (10) \qquad \left(\frac{\sigma_a}{\sigma_c^*}\right)^2 + \left(\frac{\tau_a}{\tau_c^*}\right)^2 = 1 \qquad (11)$$

For **non-proportional loading process** (the ratio of normal and shear stress components varies during the loading process, the loading trajectory is curvilinear in the graph) the factor of safety cannot be calculated using the above formula; similarly to the approach applied in Haigh diagram, it should be determined from the graphical representation as a ratio of the lengths of overloading and loading trajectories. The approach can be applied also for non-symmetric cycles (then equivalent stress amplitude is used) or even for one stress component being constant (then yield stress is used for this component instead of endurance limit).

Algorithm of assessment of factor of safety in the concept of nominal stresses (valid for infinite life)

1. Analysis of time history of inner resultants in the bar – N(t), T(t), M_o(t), M_k(t).
2. Calculation of time history of stress components σ(t) and τ(t) in the dangerous points and determination of basic parameters of the stress cycles (σ_a, σ_m and τ_a, τ_m) for all the dangerous points.
3. Determination of the endurance limits σ_C^*, τ_C^* of the component part and creation of the respective Haigh diagrams (if mean stresses σ_m, τ_m do not equal zero).
4. Draw the operation point P into the diagram (with coordinates σ_{aP}, σ_{mP}, or τ_{aP}, τ_{mP}) and assess the loading and overloading trajectory.
5. For a **proportional loading and overloading process, the** factors of safety $k_{C\sigma}$, $k_{C\tau}$ for normal and shear stresses can be calculated separately using formulas approximating Haigh diagram (e.g. eq. (7)). Then the resulting FOS k_C (under condition of **proportionality between normal and shear components of stresses**) can be calculated using formula $$k_C = \frac{k_{C\sigma} \, . \, k_{C\tau}}{\sqrt{k_{C\sigma}^2 + k_{C\tau}^2}}$$
6. For **non-proportional** loading trajectories (with a varying ratio between mean stress and stress amplitude and/or between normal and shear stresses) the FOS should be assessed on the basis of the corresponding graph (σ_m-σ_a, τ_m-τ_a, τ_a-σ_a); the limit values can be calculated as coordinates of points of intersection between the loading (or overloading) trajectory and the limit envelope.
7. Conclusions based on the FOS value: **k_C >1 – infinite life**; k_C < 1 – finite life.
8. To reach infinite life, plastic deformations are not allowed (risk of LCF). If the stress amplitude is relatively small in comparison with midrange stress, maximum stress should be checked also using a plasticity criterion, because most simplifying approaches do not exclude plastic deformations. $$k_k = \frac{R_e}{\sigma_{red}} = \frac{R_e}{\sqrt{\sigma_h^2 + 4\tau_h^2}}$$

Concept of local elastic stresses

This concept is based on calculation of alternating stress (amplitude, event. mean stress) in the root of the notch in the assessed part under assumption of validity of Hooke's law within all the range of loading. The amplitude of notch stress can represent two different values:

- either the stress amplitude evaluated by a theoretical calculation $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{teor}}$, which can be calculated
  - from the shape factor and amplitude of nominal stress $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{teor}} = \alpha \cdot \sigma_{\mathrm{a,nom}}$
  - using finite element method (FEM) $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{teor}} = \sigma_{\mathrm{a,MKP}}$

- or an effective stress amplitude $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{ef}}$, which characterizes the process of fatigue failure and can be evaluated
  - from the notch factor and amplitude of nominal stress $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{ef}} = \beta \cdot \sigma_{\mathrm{a,nom}}$
  - using FEM and reduction to the notch sensitivity of the material $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{ef}} = \dfrac{\beta}{\alpha} \cdot \sigma_{\mathrm{a,MKP}} = \dfrac{\sigma_{\mathrm{a,MKP}}}{n_{\mathrm{G}}}$

Here the ratio $n_{\mathrm{G}} = \dfrac{\alpha}{\beta} > 1$ represents a supporting influence of stress gradient and can be assessed for most steels on the basis of a relative stress gradient (see Shigley, Chapter 6, for details).

Then **fatigue failure occurs if it holds** $\left(\sigma_{\mathrm{a,vr}}\right)_{\mathrm{ef}} = \sigma'_{\mathrm{C}}$

Corrections of endurance limit for concept of local elastic stresses

The corrected endurance limit of the component part under completely reversed cycle σ_C' (for $\sigma_m = 0$) can be determined from the fatigue strength σ_{Co} obtained under experimental testing in bending[1]:

$$\boxed{\sigma_C' = k_a k_b k_c k_d k_e k_f \, \sigma_{Co}} \qquad (12)$$

where k_a = surface factor; k_b = size factor
k_c = load factor; k_d = temperature factor
k_e = reliability factor (considers dispersion of experiments); k_f = factor for other influences

Main differences in comparison with concept of nominal stresses:

1. Endurance limit is not reduced by the notch factor – it is included in stress calculation.
2. Basic endurance limit is evaluated in flexion (not in tension-compression).
3. Differences in correction factors (influence of temperature, reliability...).

Mutual recalculation of endurance limits between both concepts: $\sigma_C' = \sigma_C^* . \beta$

and recalculation of endurance limits between tension and bending $\sigma_C' = \sigma_{Co} . k_{C-tension}$

$k_{C\text{-tension}}$ – load factor for tension in the local elastic stress concept, basic assessment may be $k_{C\text{-tension}} \approx 0,85$

It holds for symmetric (completely reversed) torsion: $\tau_C' = k_C \sigma_{Co} \approx 0,59 \sigma_{Co}$ (otherwise similar to σ_C').

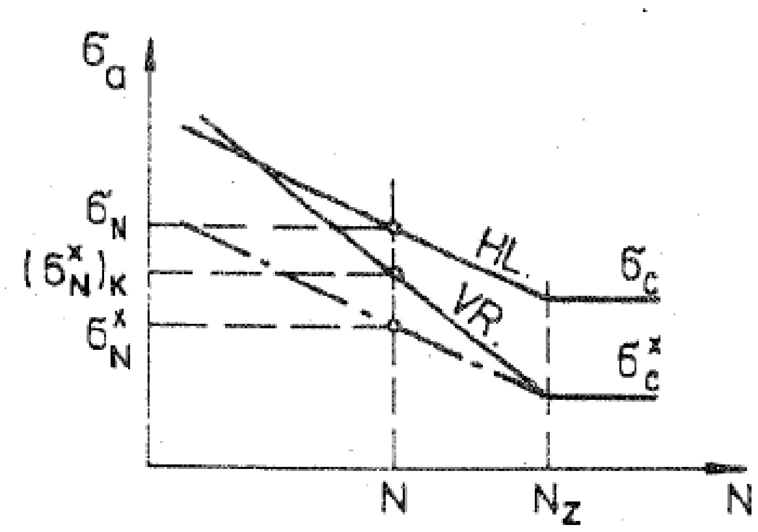
[1]according Shigley, or Marin, J.: *Mechanical Behavior of Engineering Materials*. Englewood Cliffs, Prentice-Hall 1962.

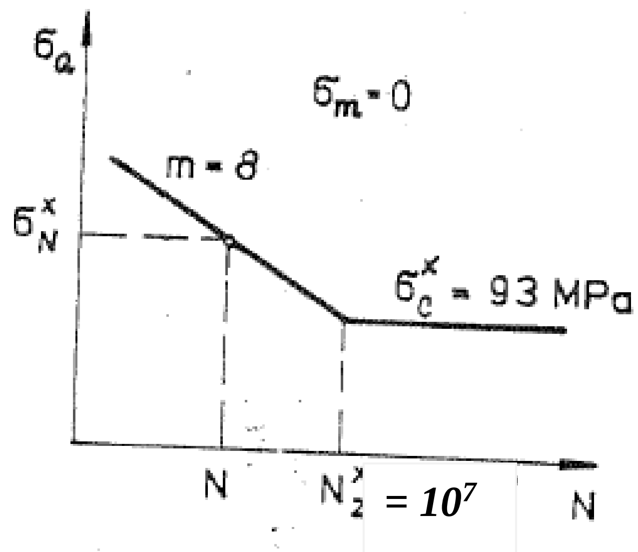
Approaches for finite life in HCF region

1a) Under constant stress amplitude and completely reversed cycle

Equation (2) of the ramped part of S-N curve[5] $$\left(\sigma_{\mathrm{N}}^{\mathrm{x}}\right)^{\mathrm{m}} \cdot N = \left(\sigma_{\mathrm{C}}^{\mathrm{x}}\right)^{\mathrm{m}} \cdot N_{\mathrm{z}}^{\mathrm{x}}$$

If we know, in addition to the endurance limit, another number N of cycles to fracture for the stress amplitude σ_N^x, we are able to calculate the exponent m. However, due to non-parallelism of the limit S-N lines, the notch factor is not constant and the value for infinite life should not be used generally.





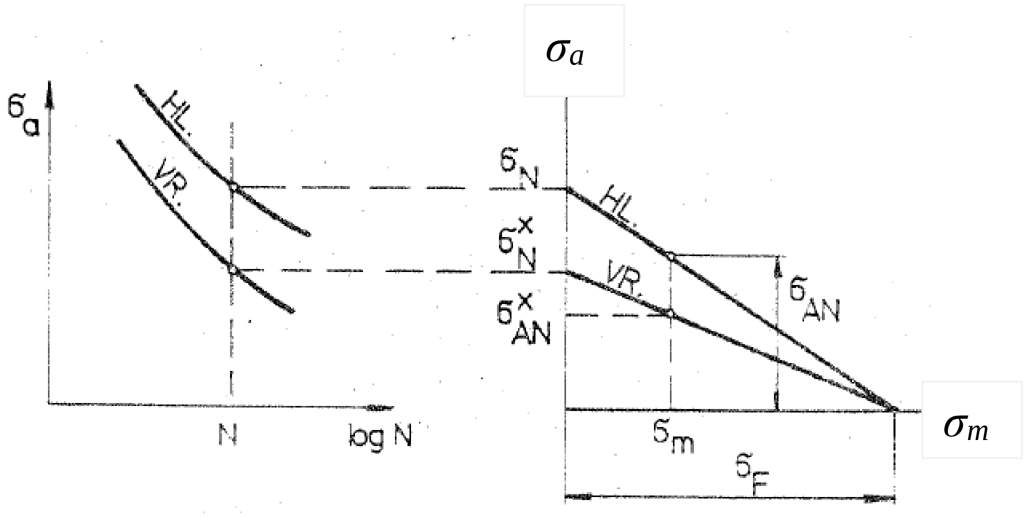
[5] Both S-N diagrams are in logarithmic scales.

1b) Procedure for a non-symmetric stress cycle

In HCF region it is possible to transform a non-symmetric cycle (with stress amplitude σ_{AN} and $\sigma_m > 0$) into a fictitious symmetric one (with amplitude σ_N) showing the same factor of safety; the procedure is based on simplified Haigh diagrams, similarly to the concept of nominal stresses for life.
For instance, when using Serensen simplification of Haigh diagram similar to eq. (7) and (8), we can obtain the following formulas for amplitude σ_N of the fictitious symmetric cycle (see figure below):

$$\sigma_N = \sigma_{AN} + \psi_\sigma \sigma_m$$ for a smooth (unnotched) specimen (HL.) and

$$\sigma_N^x = \sigma_{AN}^x + \frac{\sigma_{AN}^x}{\sigma_{AN}} \psi_\sigma \sigma_m$$ for a notched specimen (VR.).



2) Loading with varying stress amplitude (uniaxial stress state)

Procedure for several different stress amplitudes – hypothesis of linear accumulation of damage (Palmgren-Miner)

This approach is based on the assumption that partial damage ΔD_i caused by a family of loading cycles (with approx. identical parameters) is given by the ratio of the number of cycles n_i in this family to the number of cycles till failure $N_{f\,i}$ valid for this family of cycles:

$$\Delta D_i = \frac{n_i}{N_{f\,i}} \tag{13}$$

Fatigue failure occurs when the total damage by all families of loading cycles D_{Nf} reaches the limit value c. This value should be determined on the basis of experiments; if these are not available, theoretical value of $c=1$ can be used. The total damage D_{Nf} is calculated as summation of partial damages caused by individual families of cycles; the failure criterion can be formulated as follows:

$$D_{Nf} = \sum_{i=1}^{s} \Delta D_i = \sum_{i=1}^{s} \frac{n_i}{N_{fi}} = c$$

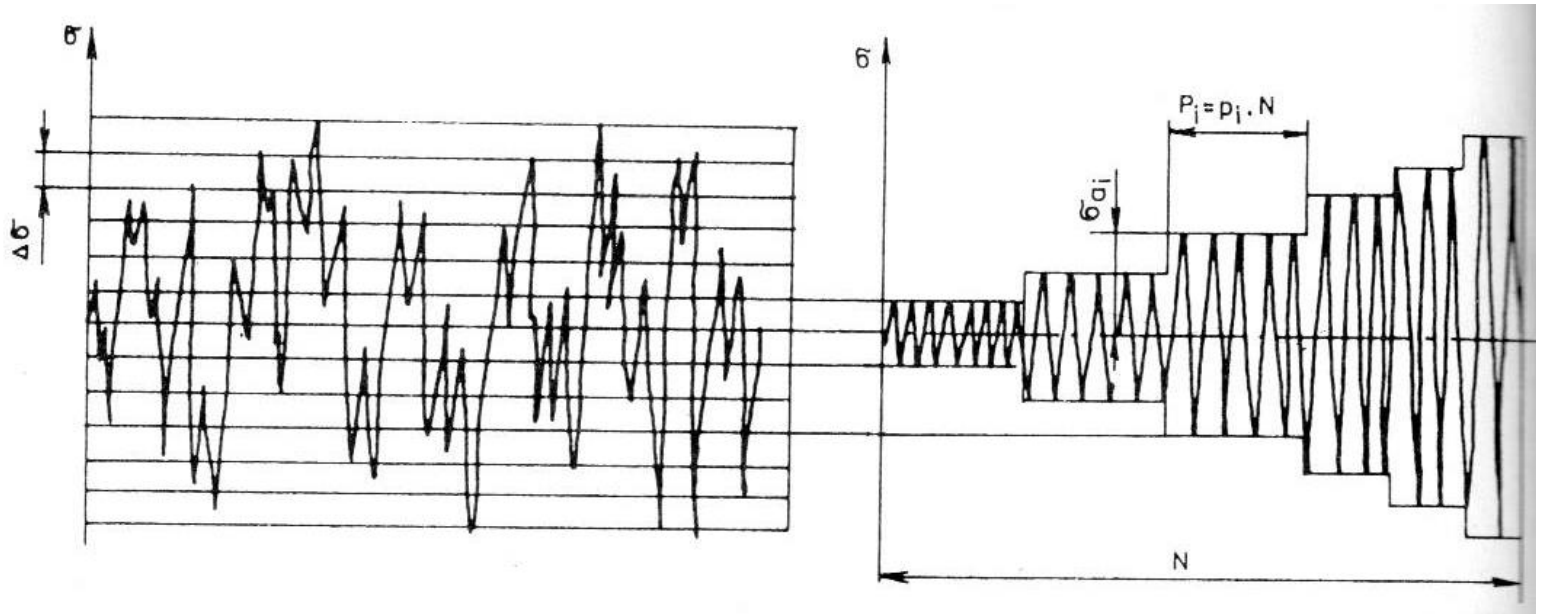
where s represents the total number of families of loading cycles.
After substituting the approximation of S-N curve by eq. (2), this formula can be manipulated to obtain the form

$$D_{Nf} = \frac{1}{N_{fz}\left(\sigma_c^*\right)^m} \sum_{i=1}^{s} \sigma_{ai}^m . n_i \tag{14}$$

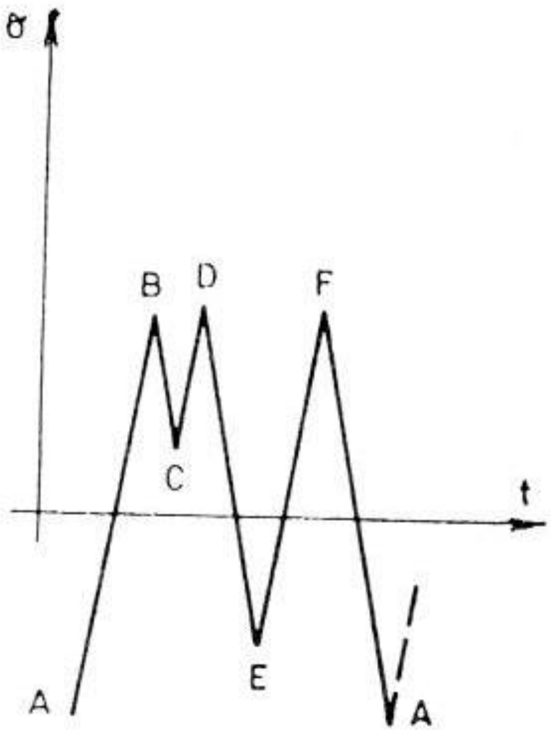
where N_{fz} is the number of cycles corresponding to the endurance limit in the S-N curve (for steels 10^7). The exponent m (being always positive) relates to the slope of the S-N curve (in high cycle region this curve appears straight in logarithmic coordinates).

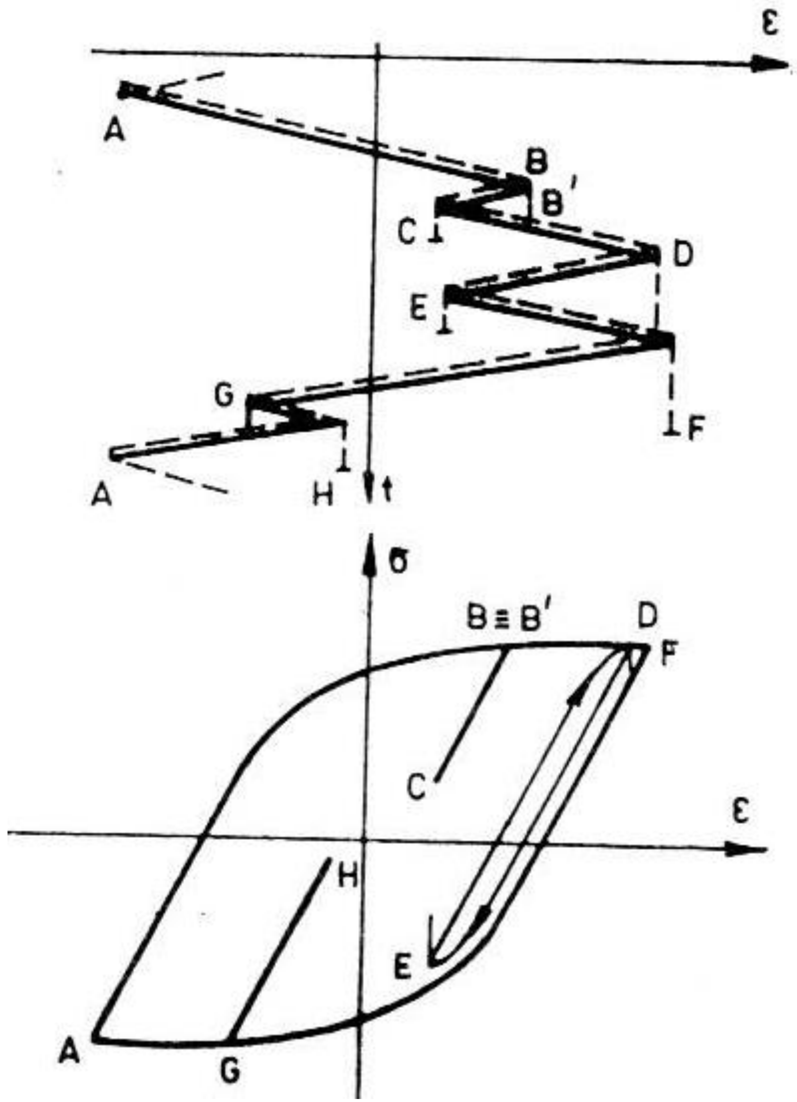
3) **Loading with stochastic stress history (in uniaxial stress state)**

Assessment of durability requires **decomposition of the stochastic loading (stress and strain) history** into individual families of loading cycles (see figure). Rainflow method and reservoir method are most frequent in this decomposition.



Rainflow method





Approaches in Low Cycle Fatigue

Concept of fictitious linear elastic stress (used in some EN or ASME standards)

It is a **highly simplified** concept based on recalculation of strains in low cycle fatigue region (i.e. above the yield limit) into stresses by using Hooke's law (Hookean, linear elastic stresses); Manson-Coffin curve defining the life for different strain amplitudes is recalculated to (fictitious) stresses in the same way. Thus we can compare these stresses instead of strains. The calculated stresses are fictitious (non-realistic) and may highly exceed not only the yield stress but also the ultimate stress (strength) of the material.
The advantage is that we can unify Manson-Coffin curve with S-N (Wöhler) curve and do not need to distinguish between LCF and HCF region.

For asymmetric cycles the following modified Manson-Coffin formula can be used:

$$\varepsilon_{at} = \frac{\sigma_f^{/} - k_m \sigma_m}{E} \left(2N_f\right)^b + \varepsilon_f^{/} \left(2N_f\right)^c ,$$

where k_m is an additional material parameter; if you do not have at your disposal experimental data on the influence of mean stress on the endurance limit of the applied material, you can use (according to Morrow) the value of $k_m = 1$.
The obtained fictitious symmetric cycle can then be evaluated using the same procedure as in the case of real symmetric cycles.

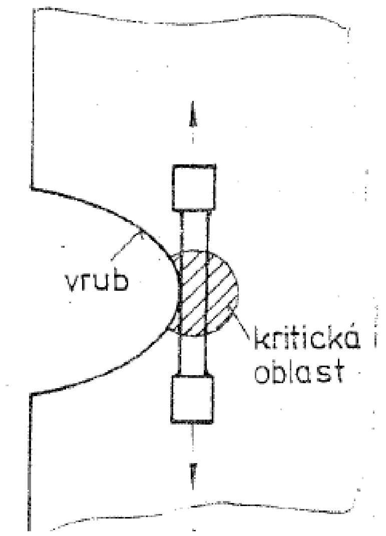
Note: *In practical applications also other approaches to evaluation of non-symmetric stress cycles are applied, e.g. SWT (Smith-Watson-Topper).*

Concepts of local elastic-plastic stresses and strains

1. Concept of plastic stress redistribution (Neuber)

For unidirectional loading Neuber derived (in 1968) the expression

$$\alpha_{\mathrm{H}} = \sqrt{\alpha_\sigma \cdot \alpha_\varepsilon} = \frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{nom}}}$$



α_{H} … shape factor – evaluated for elastic stress state under assumption of validity of Hooke’s law; (mostly denoted only α, without subscript)

σ_{H} … stress evaluated under assumption of validity of Hooke’s law within all the range of loading, denoted usually as „Hookean stress“ (or linear or elastic stress as well).

α_σ … stress concentration factor $$\alpha_\sigma = \frac{\sigma}{\sigma_{\mathrm{nom}}}$$

α_ε … strain concentration factor $$\alpha_\varepsilon = \frac{\varepsilon}{\varepsilon_{\mathrm{nom}}}$$

σ … true stress in the root of the notch (local stress in the root of the notch – local true notch stress)

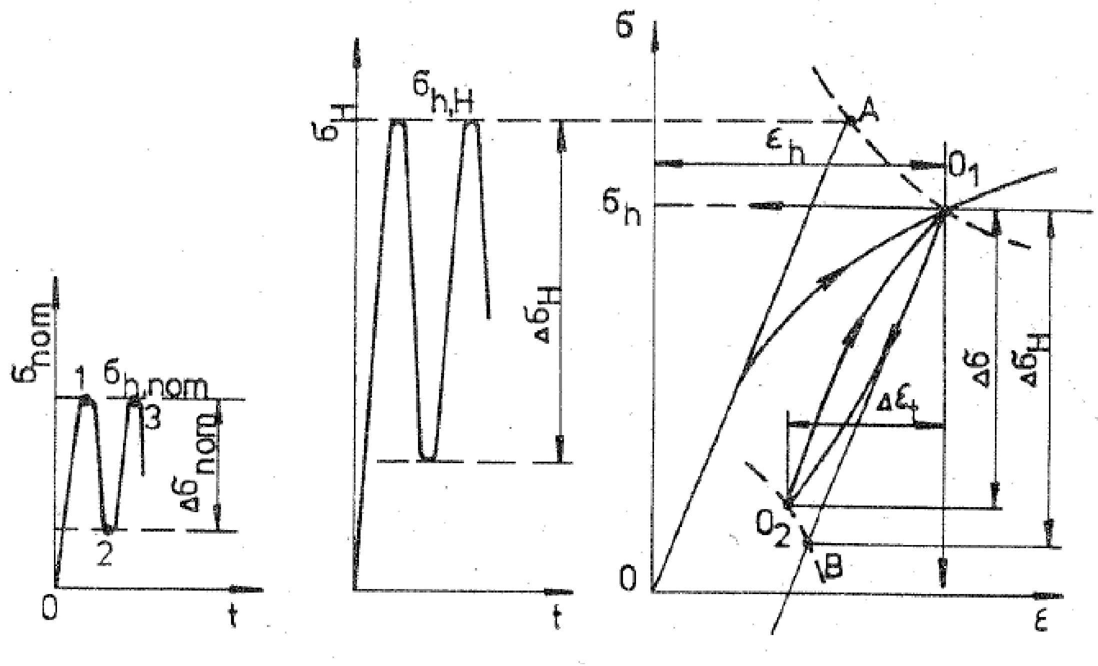
After substitution
$$\frac{\sigma}{\sigma_{\text{nom}}} \cdot \frac{\varepsilon}{\varepsilon_{\text{nom}}} = \alpha^2$$

or
$$\sigma \cdot \varepsilon = \alpha^2 \cdot \sigma_{\text{nom}} \cdot \varepsilon_{\text{nom}} = \frac{\left(\alpha \cdot \sigma_{\text{nom}}\right)^2}{E} = \frac{\sigma_{\text{H}}^2}{E} = konst. \quad => \sigma.\varepsilon = const. \tag{15}$$

We have obtained equation describing an equiaxed hyperbole in coordinate system $\sigma - \varepsilon$.

Note: *Although not emphasized, it was assumed that nominal stresses and strains are purely elastic here, given by Hooke's law:* $\varepsilon_{\text{nom}} = \dfrac{\sigma_{\text{nom}}}{E}$

The above formulas have been derived for monotonically increasing load. Later it was confirmed experimentally that they can be applied also for a cyclic load.



a) Opening halfcycle (O – 1)

This halfcycle represents unidirectional loading from the origin 0 to point 1. We are seeking for the point of intersection of the equiaxed hyperbole with the cyclic stress-strain curve.

- Equation of the hyperbole with origin in point 0: $$\sigma_{\mathrm{h}} \cdot \varepsilon_{\mathrm{h}} = \frac{\left(\sigma_{\mathrm{h,nom}} \cdot \alpha_{\mathrm{H}}\right)^2}{E} = \frac{\sigma_{\mathrm{hH}}^2}{E}$$

- Equation of the cyclic stress-strain curve: $$\varepsilon_{\mathrm{h}} = \frac{\sigma_{\mathrm{h}}}{E} + \left(\frac{\sigma_{\mathrm{h}}}{K'}\right)^{1/n'}$$

  By solving these two equations we obtain maximum stress σ_{h} and maximum strain ε_{h} .

b) Consecutive halfcycle (1 – 2)
This halfcycle represents unloading from point 1 to point 2.
We are seeking for the point of intersection of the equiaxed hyperbole with its origin in point O_1 with the respective branch of hysteresis loop.

- Equation of the hyperbole with origin in point O_1 : $$\Delta\sigma \cdot \Delta\varepsilon = \frac{\left(\alpha_{\mathrm{H}} \cdot \Delta\sigma_{\mathrm{nom}}\right)^2}{E} = \frac{\Delta\sigma_{\mathrm{H}}^2}{E}$$

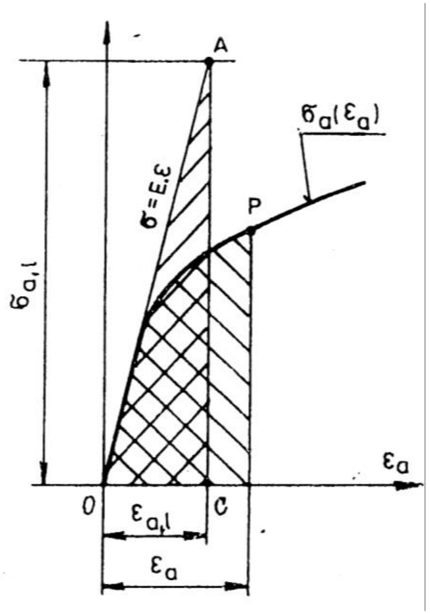
- Equation of the branch of hysteresis loop: $$\Delta\varepsilon_{\mathrm{t}} = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'}$$

  By a (numerical) solution to these two equations we obtain stress range $\Delta\sigma$ and range of total strain $\Delta\varepsilon_{\mathrm{t}}$, eventually amplitude of total strain ε_{at}. Then we can assess the number of cycles till failure on the basis of Manson-Coffin curve. This assessment is usually conservative when compared with reality.

  ***Note.**: More than ten different modifications of the original Neuber concept have been proposed till now.*

2. Concept of equivalent energy (energy criterion Molski – Glinka)

It is based on the assumption of the same value of isovolumic strain energy density (energy of the shape change) for linear elastic and elastic-plastic deformations.



$$\varepsilon_{\mathrm{h}} = \frac{\sigma_{\mathrm{h}}}{E} + \left(\frac{\sigma_{\mathrm{h}}}{K'}\right)^{1/n'}$$

$$\frac{1}{2}\sigma_{a,l}.\varepsilon_{a,l} = \int_0^{\varepsilon_a} \sigma_a.d\varepsilon_a$$

Calculation of ε_a can be carried out by using numerical methods of solution.
The strain amplitude values calculated using Molski-Glinka criterion are lower than using Neuber approach, it means we get higher numbers of cycles and a better agreement with reality.

3. Using FEM in evaluation of elastic-plastic deformations

Finite element method (FEM) enables us to calculate the strain amplitude for any shape of the body and elastic-plastic material model. Maximum value (mostly in a notch) can then be used for assessment of the number of cycles till failure in Manson-Coffin curve.

All the presented concepts for finite life presented here are valid for uniaxial stress states (and for non-symmetric stress-strain cycles). They can be formulated also for biaxial stress states, namely for both proportional and non-proportional loading.

Fatigue for multiaxial stress states (and asymmetric cycles)

There are not yet general rules or standards for evaluation in this field.

Actual information and databases can be found e.g. at the following web pages:

www.pragtic.com
www.fatiguecalculator.com
www.freewebs.com/fatigue-life-integral/