evropský sociální fond v ČR EVROPSKÁ UNIE MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

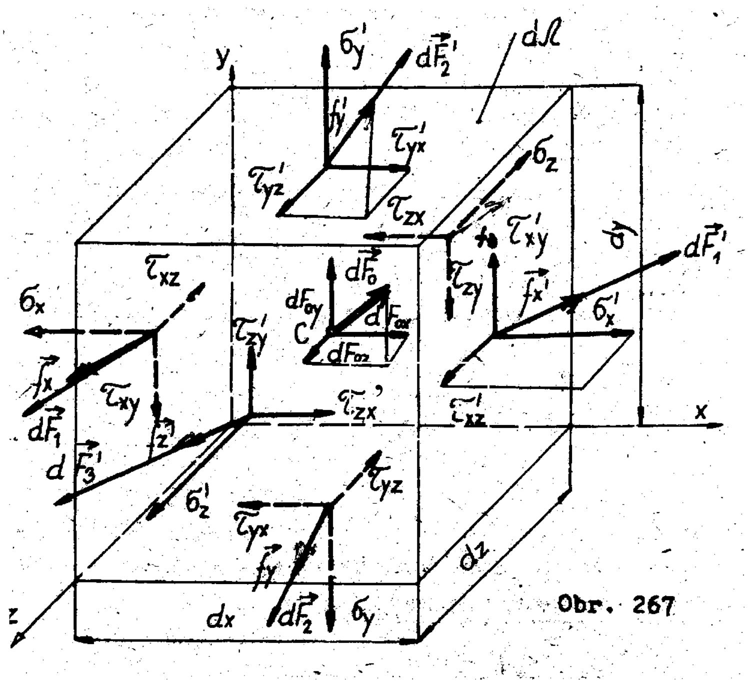
System of general equations of elasticity

General Hooke's law

In the course Strength of materials I, problems of elasticity of bars only were solved. To solve stresses and strains in a general 3D body, it is necessary to create and solve a system of general equations of elasticity. Their solution in a specific case can be found by means of two basic approaches:

- **differential approach** – solution to a system of differential equations,
- variation approach – formulation of an energetic quantity and finding its minimum by means of **calculus of variations**.

Material is assumed to be a homogeneous isotropic linear elastic continuum; its mechanical properties can be described by global elastic parameters (E, μ – characterizing its elastic behaviour) and by other material parameters (R_e – yield stress or $R_{p0,2}$ – conventional yield stress, R_m – ultimate stress, σ_C – fatigue strength, K_{IC} – fracture toughness, etc.) characterizing conditions of failure. Three of the necessary differential equations are formulated on the basis of equilibrium of a threefold infinitesimal element.



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + o_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + o_y = 0 \qquad \vec{o} = \vec{A}\rho$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + o_z = 0$$

These equations are often called **Cauchy equations of equilibrium**.

Overview of the most important **properties** of the stress **tensor** (revision from the course of Strength of materials I).
The listed properties are explained or specified in greater detail at the following pages. They are common for any tensor quantities.

1. It can be expressed in the form of a **square matrix T_σ** (3 x 3 in a 3D space).
2. Coordinates of the tensor in any rotated coordinate system can be calculated from this matrix; they represent **stress components** acting in the corresponding planes.
3. A position can be found among the rotated coordinate systems (different rotated positions of the element) in which all shear stresses equal zero; the corresponding normal stresses are then called **principal stresses**.
4. Principal directions (directions of the principal stresses) are **mutually perpendicular**; the angle between two principal directions is arbitrary only in the case that the corresponding two principal components equal in magnitude (in the figure $\sigma_2 = \sigma_3$).
5. Principal stresses can be calculated from the **characteristic equation** of the stress tensor; it is a third order algebraic equation having three real solutions. The coefficients in this equation are **invariants** of the stress tensor.
6. A graphical representation of the stress tensor in **Mohr's plane** represents stress components in all the rotated coordinate systems and enables us to calculate minimum and maximum values of the normal and shear stresses easily.
7. If some of the stress components equal zero or each other, **specific types of stress states** can be defined: biaxial, uniaxial, equibiaxial, hydrostatic, shear, or bar-type stress states.
8. Any stress tensor can be decomposed into its **spherical** K_σ and **deviatoric** D_σ parts.

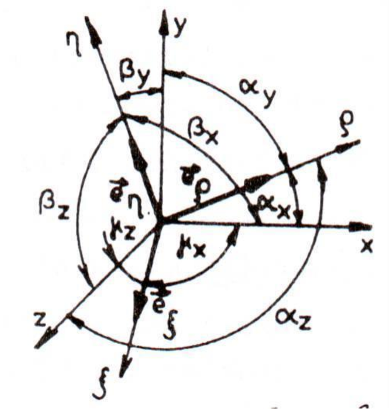
1)

$$T_\sigma = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

3)

$$T_\sigma = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

2)



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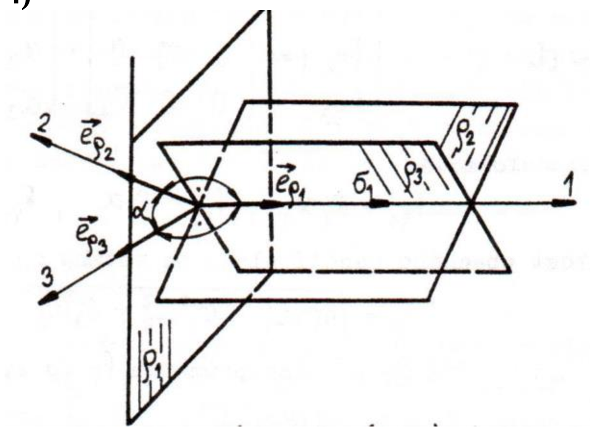


$$\sigma_\rho = \alpha_\rho^T.T_\sigma.\alpha_\rho \qquad \alpha_\rho = \begin{bmatrix} \cos\alpha_x \\ \cos\alpha_y \\ \cos\alpha_z \end{bmatrix}$$

5) $\sigma^3 - I_1.\sigma^2 + I_2.\sigma - I_3 = 0$

$$\sigma_\eta = \alpha_\eta^T.T_\sigma.\alpha_\eta$$

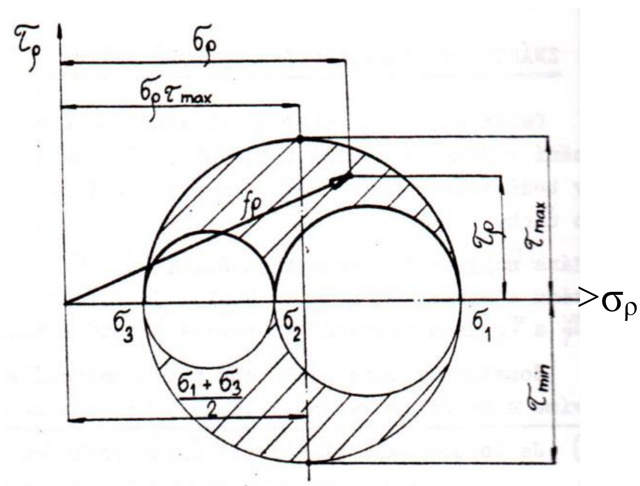
$$\sigma_\xi = \alpha_\xi^T.T_\sigma.\alpha_\xi \qquad \alpha_\eta = \begin{bmatrix} \cos\beta_x \\ \cos\beta_y \\ \cos\beta_z \end{bmatrix}$$

$$\tau_{\rho\eta} = \alpha_\rho^T.T_\sigma.\alpha_\eta$$

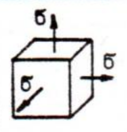
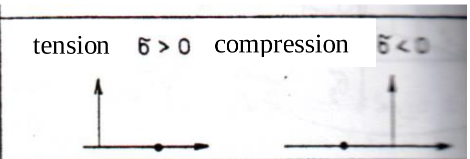
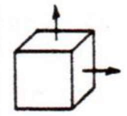
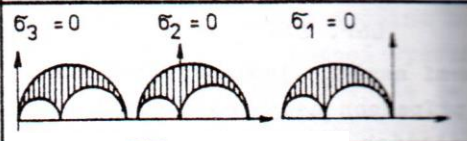
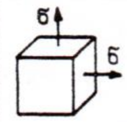
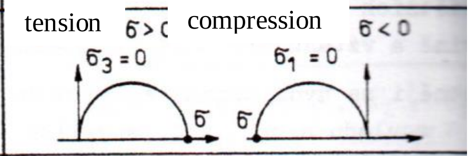
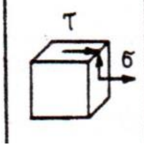
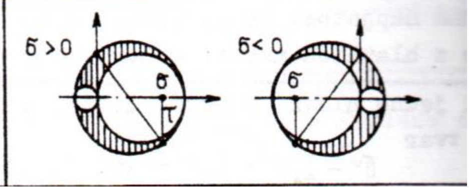
$$\tau_{\rho\xi} = \alpha_\rho^T.T_\sigma.\alpha_\xi$$

$$\tau_{\eta\xi} = \alpha_\eta^T.T_\sigma.\alpha_\xi \qquad \alpha_\xi = \begin{bmatrix} \cos\gamma_x \\ \cos\gamma_y \\ \cos\gamma_z \end{bmatrix}$$

6)



7) Specific types of stress states

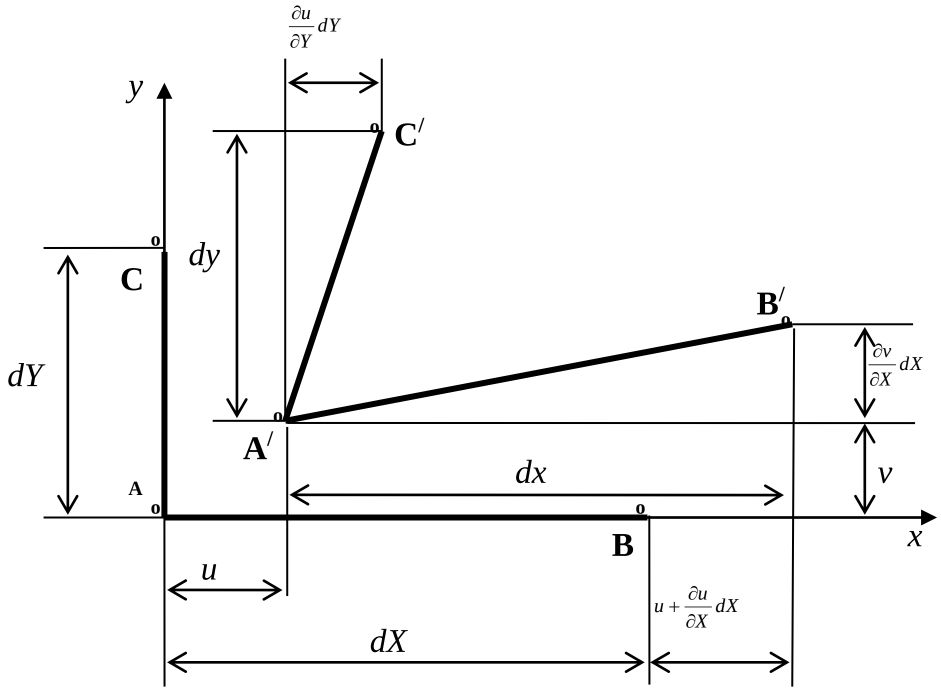
3D	**iso-tropic**		all principal stresses equal in magnitude $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$	
2D	**gene-ral**		one principal stress equals zero, two non-zero principal stresses are not equal	
	iso-tropic (equi-biaxial]		one principal stress equals zero, two non-zero principal stresses are equal	
	bar-type		stress state is determined by stress components in one wall of the element $\sigma_{1,3} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$	

8)

$$T_\sigma = D_\sigma + K_\sigma$$

$$D_\sigma = \begin{bmatrix} \sigma_x - \sigma_s & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_s & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_s \end{bmatrix} \qquad K_\sigma = \begin{bmatrix} \sigma_s & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_s \end{bmatrix}$$

Strain tensor
strain-displacement (geometrical) equations



$$\varepsilon_x = \frac{\partial u}{\partial x}; \qquad \varepsilon_y = \frac{\partial v}{\partial y}; \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Properties of the strain tensor

can be formulated on the basis of tensor calculus (analogy with stress tensor):

1. It can be expressed in the form of a **square matrix $\mathbf{T}_\varepsilon$** (3 x 3 in a 3D space).
2. Coordinates of the tensor in any rotated coordinate system can be calculated from this matrix; they represent **strain components** in the corresponding planes.
3. A position can be found among the rotated coordinate systems (different rotated positions of the element), in which all the three angular strains equal zero; the corresponding length strains are then called **principal strains**.
4. Principal directions (directions of the principal strains) are **mutually perpendicular**; the angle between two principal directions is arbitrary only in the case that the corresponding two principal components equal in magnitude (in the figure $\varepsilon_2 = \varepsilon_3$).
5. Principal strains can be calculated from the **characteristic equation** of the strain tensor; it is a third order algebraic equation having three real solutions. The coefficients in this equation are **invariants** of the strain tensor.
6. A graphical representation of the strain tensor in **Mohr's plane** represents strain components in all the rotated coordinate systems and enables us to calculate minimum and maximum values of the length and angular strains easily.
7. If some of the strain components equal zero or each other, **specific types of strain states** can be defined: biaxial, equibiaxial and other strain states.
8. Any strain tensor can be decomposed into its spherical (**volumetric**) K_ε and **deviatoric** (shape) D_ε parts.

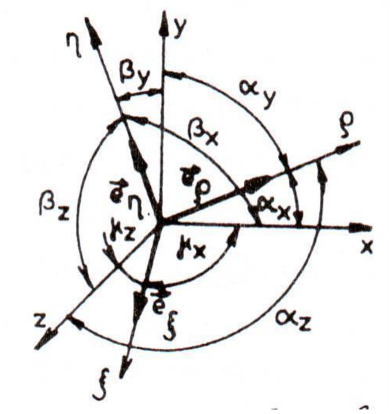
1)

$$T_\varepsilon = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z \end{bmatrix}$$

3)

$$T_\varepsilon = \begin{bmatrix} \varepsilon_I & 0 & 0 \\ 0 & \varepsilon_{II} & 0 \\ 0 & 0 & \varepsilon_{III} \end{bmatrix}$$

2)



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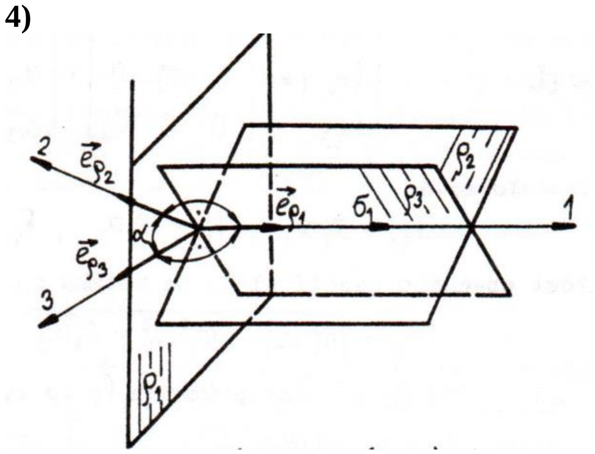


$$\varepsilon_\rho = \alpha_\rho^T . T_\varepsilon . \alpha_\rho \qquad \alpha_\rho = \begin{bmatrix} \cos\alpha_x \\ \cos\alpha_y \\ \cos\alpha_z \end{bmatrix}$$

5) $\varepsilon^3 - E_1.\varepsilon^2 + E_2.\varepsilon - E_3 = 0$

$$\varepsilon_\eta = \alpha_\eta^T . T_\varepsilon . \alpha_\eta$$

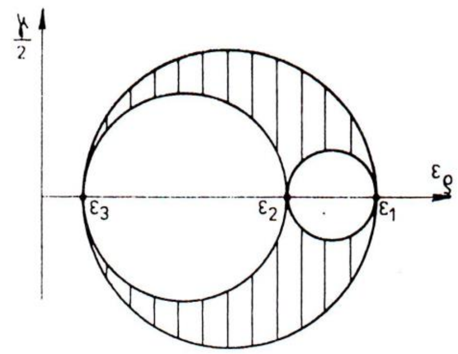
$$\varepsilon_\xi = \alpha_\xi^T . T_\varepsilon . \alpha_\xi \qquad \alpha_\eta = \begin{bmatrix} \cos\beta_x \\ \cos\beta_y \\ \cos\beta_z \end{bmatrix}$$

$$\frac{\gamma_{\rho\eta}}{2} = \alpha_\rho^T . T_\varepsilon . \alpha_\eta$$

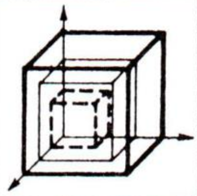
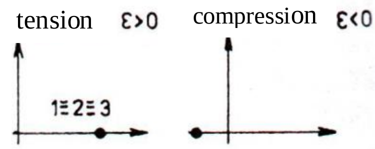
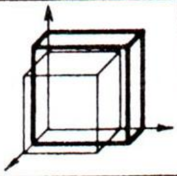
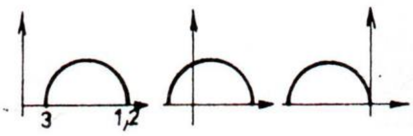
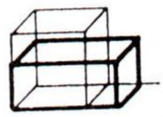
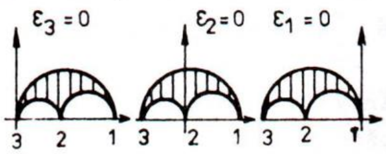
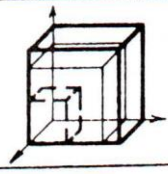
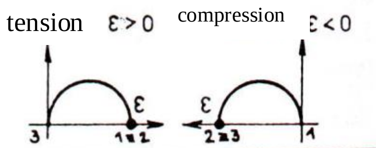
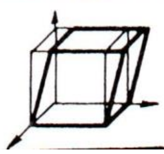
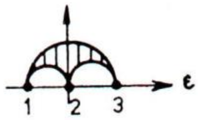
$$\frac{\gamma_{\rho\xi}}{2} = \alpha_\rho^T . T_\varepsilon . \alpha_\xi \qquad \alpha_\xi = \begin{bmatrix} \cos\gamma_x \\ \cos\gamma_y \\ \cos\gamma_z \end{bmatrix}$$

$$\frac{\gamma_{\eta\xi}}{2} = \alpha_\eta^T . T_\varepsilon . \alpha_\xi$$

6)



7) Strain state types

3D	**isotropic**		all principal strains equal in magnitude $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$	
	semi-isotropic		two principal strains equal in magnitude	
2D	**general**		one principal strain equals zero, two non-zero principal strains are not equal	
	isotropic (equibiaxial)		two non-zero principal strains are equal	
	shear		shear $\varepsilon_1 = -\varepsilon_3 = \frac{\gamma}{2}, \varepsilon_2 = 0$	

8) Strain tensor T_ε can be decomposed into its volumetric K_ε and deviatoric D_ε parts.

$$T_\varepsilon = K_\varepsilon + D_\varepsilon$$

Mean strain $\varepsilon_s = \dfrac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$

$$K_\varepsilon = \begin{bmatrix} \varepsilon_s & 0 & 0 \\ 0 & \varepsilon_s & 0 \\ 0 & 0 & \varepsilon_s \end{bmatrix} \qquad D_\varepsilon = \begin{bmatrix} \varepsilon_x - \varepsilon_s & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y - \varepsilon_s & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z - \varepsilon_s \end{bmatrix}$$

Basic types of general equations of elasticity:

1. Cauchy equations of statical equilibrium of an infinitesimal element:

a) Inner element:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+o_x=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+o_y=0$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+o_z=0$$

The element shape is a hexahedron (or cube), the specific volumetric load $\vec{o}$ in the equations can be a gravitational, centrifugal, electromagnetic or other force.

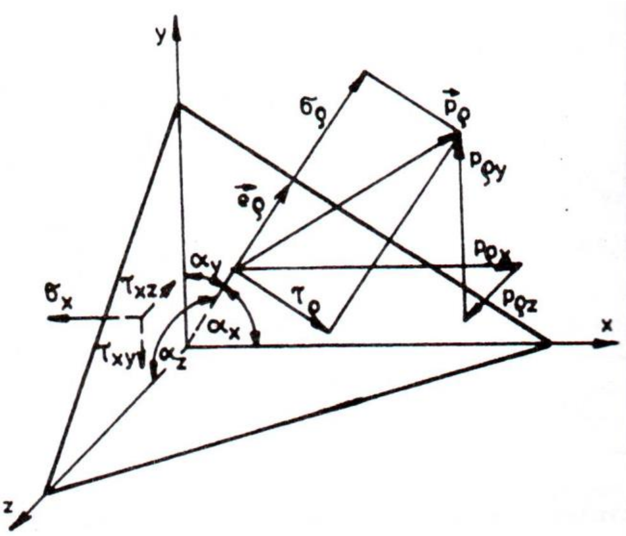
b) Boundary element:

$$p_{\rho x}=\sigma_x.\cos\alpha_x+\tau_{yx}.\cos\alpha_y+\tau_{zx}.\cos\alpha_z$$

$$p_{\rho y}=\tau_{xy}.\cos\alpha_x+\sigma_y.\cos\alpha_y+\tau_{yz}.\cos\alpha_z$$

$$p_{\rho z}=\tau_{xz}.\cos\alpha_x+\tau_{yz}.\cos\alpha_y+\sigma_z.\cos\alpha_z$$

Note: In textbooks the cos symbol is often omitted in these equations.



The element shape is a tetrahedron with one wall being on the body surface. If the body surface is unloaded, the corresponding **traction**[1] **p** equals zero.

2. Strain-displacement equations:

The above partial differential equations, relating components of the displacement vector with components of the strain tensor.

3. Constitutive equations:

They express mutual relations among the components of stress and strain tensors. In the linear elasticity, these relations are described by the **general Hooke's law**.

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[1] Traction (or more rigorously Cauchy traction vector) is infinitesimal force per infinitesimal area of an imaginary separating surface. Mostly it is used for body surface or interface between bodies, to distinguish the stress acting here and having character of a vector from the stress inside the body defined uniquely by stress tensor.

General (3-dimensional) Hooke’s law

This law is valid for a homogeneous isotropic linear elastic material (Hookean material).
It can be derived easily (using the principle of superposition) on the basis of uniaxial loads in three mutually perpendicular directions. For explicitly expressed strain components it holds:

$$\varepsilon_x = \frac{1}{E}.\left[\sigma_x - \left(\mu\sigma_y + \mu\sigma_z\right)\right] \qquad \gamma_{xy} = \frac{1}{G}.\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}.\left[\sigma_y - \left(\mu\sigma_x + \mu\sigma_z\right)\right] \qquad \gamma_{xz} = \frac{1}{G}.\tau_{xz}$$

$$\varepsilon_z = \frac{1}{E}.\left[\sigma_z - \left(\mu\sigma_x + \mu\sigma_y\right)\right] \qquad \gamma_{yz} = \frac{1}{G}.\tau_{yz}$$

Inverse relations (with explicitly expressed stresses) can be derived e.g. using Cramer’s rule: (the example below holds for one of the normal stress components):

$$\sigma_x = \frac{E}{1+\mu}.\left[\varepsilon_x + \frac{\mu}{1-2\mu}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\right]$$

By introducing new elastic parameters G and λ (**Lamé constants**) we obtain:

$$\sigma_x = 2G.\varepsilon_x + \lambda e \qquad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G.\varepsilon_y + \lambda e \qquad \tau_{xz} = G\gamma_{xz}$$

$$\sigma_z = 2G.\varepsilon_z + \lambda e \qquad \tau_{yz} = G\gamma_{yz}$$

where $G = \dfrac{E}{2(1+\mu)}$, $\quad \lambda = \dfrac{E\mu}{(1+\mu)(1-2\mu)}$,

Relative volume change $\quad e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

This relation is **limited to strains ε_i < 1%**!.

Another dependent elastic parameter is bulk modulus, defined as ratio of mean stress σ_S to relative volumetric change e

$$K = \frac{\sigma_s}{e} = \frac{\dfrac{\sigma_x + \sigma_y + \sigma_z}{3}}{\dfrac{\Delta V}{V}} \left[Pa\right]$$

and related to the other elastic constants by the following formula:

$$K = \frac{E}{3(1-2\mu)}$$

For μ=0.5, the bulk modulus tends to infinity (incompressible material).

Frequently the generalized Hooke’s law is used in its **matrix form**, or some simplified shapes of the equations can be derived, valid only for **plane stress state**, **plane strain state**, or **shear stress state**.

Hooke's law for plane (2D) stress state

Plane stress state occurs in many practical applications, e.g. at thin wall bodies (membranes, plates, discs, etc.) or in experimental stress evaluation (strains are dominantly measured on the surface where plane stress state occurs even in general bodies).

If $\sigma_z = 0$ in a plane stress state, then simplified equations can be derived for the other normal stress components:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right)$$
$$\sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right)$$

Equations relating shear stresses with angular strains remain unchanged.

For **plane strain state** it can be shown that a **3D stress state** occurs.

The only exception is **shear strain state** (a specific type of plane strain state with $\varepsilon_1 = -\varepsilon_3$) in which also a specific type of plane stress state occurs (**shear stress state** with $\sigma_1 = -\sigma_3$).

Elastic strain energy *W* (potential energy of the elastic deformation) can be obtained by integration of the strain energy density Λ throughout the volume of the body. For multiaxial state of stress the strain energy density is given by summation of works done by all the stress components; in a principal coordinate system summation of works done by principal stresses is sufficient and it holds (for a Hookean material):

$$\Lambda = \sum_{i=1}^{3} \int \sigma_i d\varepsilon_i = \frac{1}{2}\sum_{i=1}^{3} \sigma_i \varepsilon_i$$

$$W = \int_V \Lambda dV$$

Boundary conditions (BCs) are necessary for any solution to differential equations. There are two basic types of BCs:

- deformation BC – (known displacement values prescribed in some points of the body)
- force BC – a pressure is prescribed on a part of the body surface. This BC is valid also for any free surface of the body (pressure equals zero).

Review:

Output values

- displacements (displacement vector with its components *u*, *v*, *w*),
- strains (strain tensor T_ε with its independent components ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{xz}),
- stresses (stress tensor T_σ with its independent components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz}),

In total 15 unknown components are described with functions u(x,y,z), T_ε(x,y,z), T_σ(x,y,z).

These functions of the output values are to be solved analytically from the system of equations of general elasticity, consisting of the following 15 equations:

- equations of static equilibrium of a 3D element – 3 partial differential equations,
- strain-displacement equations – 6 partial differential equations,
- Hooke's law – 6 linear algebraic equations.

Fundamental problem of the general theory of elasticity

Its basic formulation is in the form of the so called **direct problem** (inputs $\rightarrow$ outputs).

Inputs: geometry, material, supports, loads.
Outputs: displacements, stresses, strains.

Kirchhoff has proven the uniqueness of the solutions to direct problems in the theory of elasticity.

Inverse (indirect) problem (output $\rightarrow$ input)**:** on the basis of a known output parameter (e.g. allowable stress) some of the input parameters can be calculated (a dimension, required material strength, allowable load, etc.). Solutions to these problems are not unique, the procedures can be numerically unstable (ill conditioned).

Optimization problem: input parameters are varied with the aim to achieve an extreme value of an optimization quantity (e.g. minimum weight, maximum load-bearing capacity, etc.).

Variants of solutions to the system of general equations of elasticity differ mutually by the choice of the basic unknown quantities.

- **Deformation variant:**

The procedure continues from displacements to strains and consequently to stresses. This variant is the most frequent in both analytical (based on differential calculus) and numerical (based on calculus of variations) solutions.

- **Force variant:**

The procedure continues from stresses to strains and consequently to displacements. However, with stresses as primary independent functions the continuity of displacements (and consequently continuity of the body itself) is not ensured, therefore some additional equations (**equations of compatibility**) are needed here to enforce the continuity of the body. Practical applications are very rare.

Approaches to solutions to a direct problem of general theory of elasticity

Differential approach | **Approach of variations**

Analytical solution | **Numerical solution**

Deformation variant | Force variant | Hybrid variant

Comparison of both approaches to the solution:

1. Analytical solution
Advantages: if there exists a closed-form analytical solution, functional relations among input and output quantities can be expressed explicitly; also the solutions to the inverse and optimization problems are relatively easy.
Disadvantages: analytical solutions can be found for a few problems only.

2. Numerical solution
Advantages: even very complex problems (from the point of view of geometry, material behaviour, etc.) can be solved using up-to-date computational equipment.
Disadvantages: we need an expensive software, much experience, model creation is time-consuming, we do not know any direct relations among the input and output quantities and the results can hardly be generalized; neither inverse nor optimization problems can be solved directly.

In the following chapters the **deformation variant of differential approach** is applied to find **analytical solutions** to some simplified problems. The general procedure is as follows:

strain-displacement equations $\rightarrow$ Hooke's law $\rightarrow$ Cauchy equations of equilibrium

In this way we obtain the following Lamé equations of general elasticity:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\left(1+\frac{2\mu}{1-2\mu}\right)\frac{\partial e}{\partial x}+\frac{\sigma_x}{G}=0$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\left(1+\frac{2\mu}{1-2\mu}\right)\frac{\partial e}{\partial y}+\frac{\sigma_y}{G}=0$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\left(1+\frac{2\mu}{1-2\mu}\right)\frac{\partial e}{\partial z}+\frac{\sigma_z}{G}=0$$

Their general solution is not known, they can be solved analytically only in some simplified cases.