evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Basic model bodies of general theory of elasticity

1. **General** body – analytical solution is not known.

2. **Axisymmetric** body

An axisymmetric body in the theory of elasticity must give axisymmetric results (stresses, strains, displacements); therefore it needs not only to have an axisymmetric geometry, but its material properties, supports and loads must be (approximately) axisymmetric as well. Analytical solutions are known for cylindrical and spherical (thick wall) bodies.

3. **Thin-wall** body (straightforward analytical solutions exist also in axisymmetry only)

Middle surface is curved
shell of revolution

Middle surface is planar (not curved)

in both deformed and undeformed states - **wall** (load acting only in the middle surface)

in the undeformed state, but curved in the deformed state - **plate** (load acting perpendi-culary to the middle surface)

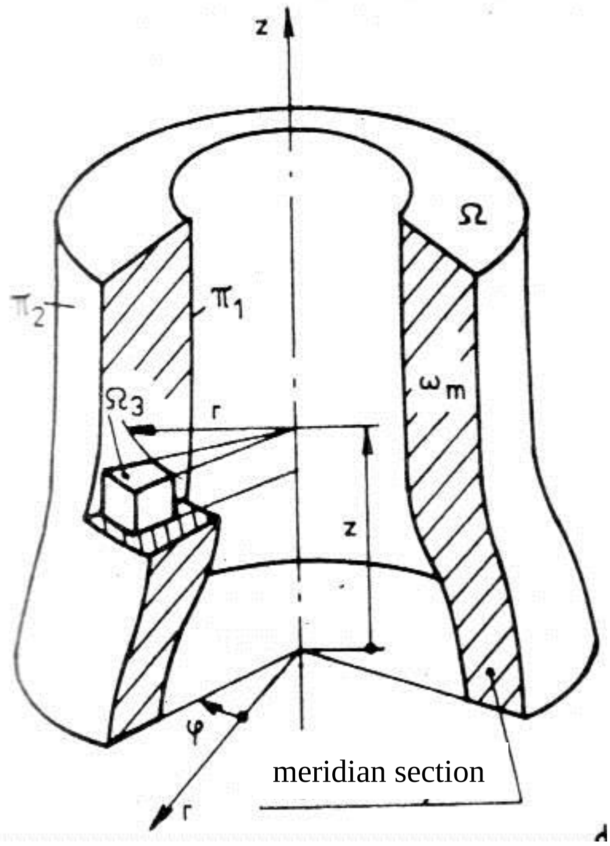
Stress is constant through the thickness - **membrane** (momentless) **shell theory**

Stress varies throughout the thickness - **moment shell theory** (bending)

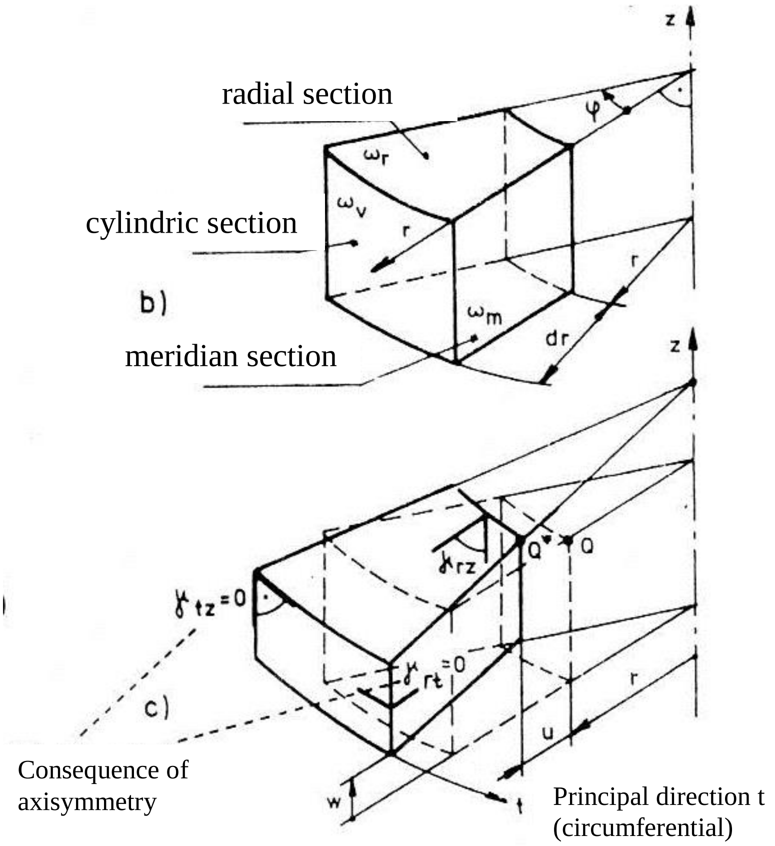
Overview of analytically solvable model bodies

- Rod-like bodies (bars, beams, columns, shafts)
- Thick-wall cylindrical and spherical body
- Rotating disc
- Axisymmetric (or rectangular) plate
- Axisymmetric membrane shell
- Cylindrical moment shell

Axisymmetric body

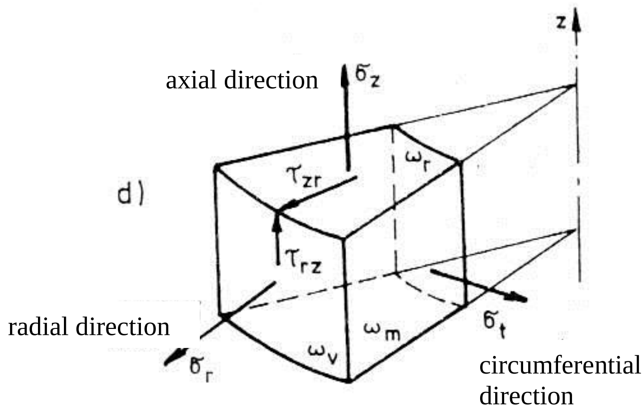


To keep the axisymmetry also for the deformed body shape, the angular strains γ_{rt} a γ_{tz} must equal zero (see the bottom fig.), therefore the t-direction (circumferential) is principal direction of the strain tensor.



To exploit advantages of axisymmetry, cylindrical coordinate system is used (instead of Cartesian) with coordinates **r, φ, z** and subscripts **r, t, z** for radial, circumferential (tangential), and axial components of the investigated quantities, respectively.

According to Hooke's law, zero angular strains correspond to **zero shear stresses** (τ_{rt} = 0 and τ_{tz} =0) and the free body diagram of an infinitesimal 3D element looks as follows:



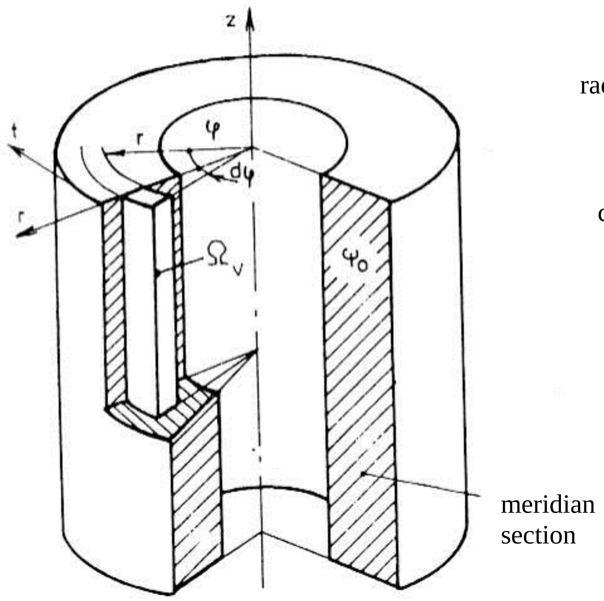
The strain and stress tensor can then be expressed by the following matrixes, each with four independent components:

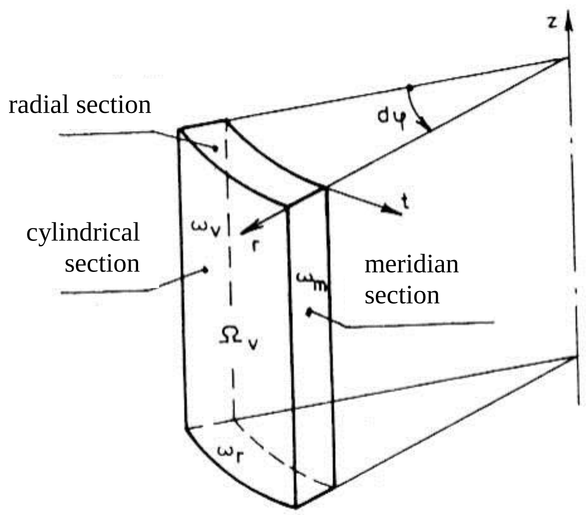
$$T_\varepsilon = \begin{bmatrix} \varepsilon_r & 0 & \gamma_{rz}/2 \\ 0 & \varepsilon_t & 0 \\ \gamma_{rz}/2 & 0 & \varepsilon_z \end{bmatrix}$$

$$T_\sigma = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_t & 0 \\ \tau_{rz} & 0 & \sigma_z \end{bmatrix}$$

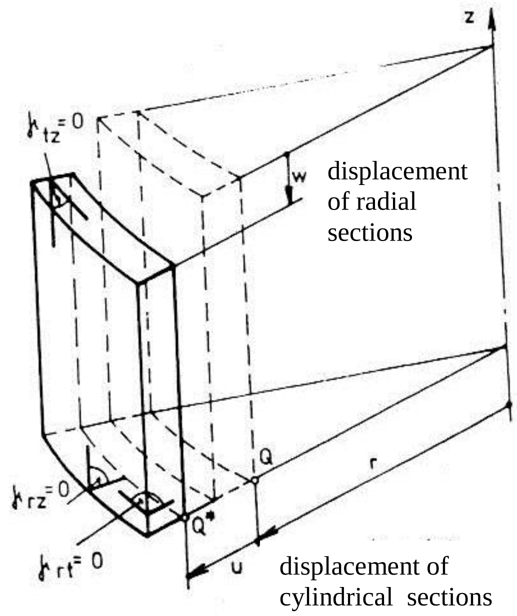
Thick-wall cylindrical body

is a special case of an axisymmetric body. At a cylindrical body neither radial section can rotate during deformation so that all the angular strains are zero, as well as all the shear stresses. Therefore the directions *r*, *t*, *z* are **principal directions** (of both stress and strain tensors).





b)



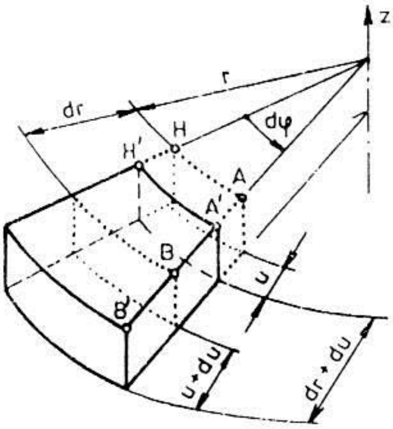
$$u = u(r) \qquad\qquad w = w(z)$$

$$\varepsilon_r = \varepsilon_r(r);\ \ \varepsilon_t = \varepsilon_t(r);\ \ \varepsilon_z = const$$

Formulation of equations used for the solution:

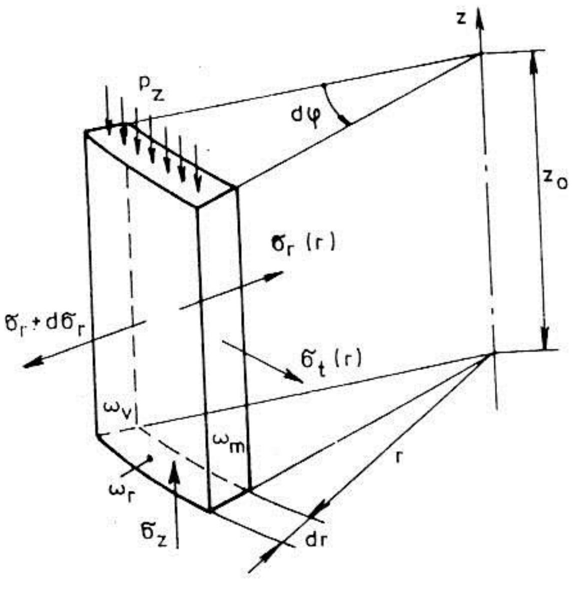
The parameters to be calculated are: $\boldsymbol{\sigma_t}$**,** $\boldsymbol{\sigma_r}$**,** $\boldsymbol{\sigma_z}$**, u,** all of them depend on the radius ***r*** only. Displacement ***w*** depends on the ***z*** coordinate only.

1. Formulation of the **strain-displacement equations** for a cylindrical coordinate system.



$$\varepsilon_r = \frac{du}{dr}; \qquad \varepsilon_t = \frac{u}{r}; \qquad \varepsilon_z = \frac{dw}{dz}$$

2. Formulation of the **equations of static equilibrium** – only the force equation for the radial direction is needed for the solution yielding eq. (1). Volumetric (gravitational) forces are neglected here.



$$\sigma_r - \sigma_t + r\frac{d\sigma_r}{dr} = 0 \qquad (1)$$

As $\varepsilon_z(r,t) = konst$, it is assumed also for the axial stresses $\sigma_z(r,t) = konst$.

3. For a **cylindric coordinate system** and with explicitly expressed stresses, **Hooke's law** can be formulated as follows:

$$\sigma_r = 2G\varepsilon_r + \lambda.e = \frac{E}{1+\mu}\left[\varepsilon_r + \frac{\mu}{1-2\mu}(\varepsilon_r + \varepsilon_t + \varepsilon_z)\right],$$

$$\sigma_t = 2G\varepsilon_t + \lambda.e = \frac{E}{1+\mu}\left[\varepsilon_t + \frac{\mu}{1-2\mu}(\varepsilon_r + \varepsilon_t + \varepsilon_z)\right]$$

and after differentiating σ_r with respect to ***r*** and substituting the strain-displacement equations we can obtain:

$$\frac{d\sigma_r}{dr} = \frac{E}{1+\mu}\left[\frac{d^2u}{dr^2} + \frac{\mu}{1-2\mu}\left(\frac{d^2u}{dr^2} + \left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) + \frac{d\varepsilon_z}{dr}\right)\right] \qquad (2)$$

By subtracting the stresses from each other we can obtain

$$\sigma_r - \sigma_t = \frac{E}{1+\mu}(\varepsilon_r - \varepsilon_t) \qquad (3)$$

and after substitution of the strain-displacement equations

$$\sigma_r - \sigma_t = \frac{E}{1+\mu}\left(\frac{du}{dr} - \frac{u}{r}\right) \qquad (4)$$

4. By substituting eqs. (2) and (4) into eq. (1) and some mathematical manipulations (taking $\frac{d\varepsilon_z}{dr} = 0$ into consideration), we can obtain the equation of static equilibrium expressed by means of the radial displacements in the following shape:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad (5)$$

5. This ordinary differential equation has its solution in the shape:

$$u = c_1 r + \frac{c_2}{r} \qquad (6)$$

6. If we return from the displacements back to stresses, we can obtain their radial and circumferential components in the following form:

$$\boxed{\sigma_r = A - \frac{B}{r^2}} \qquad (7a)$$

$$\boxed{\sigma_t = A + \frac{B}{r^2}} \qquad (7b)$$

with A and B being unknown integration constants to be evaluated from boundary conditions.

7. By substituting these results into Hooke's law in the form:

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_r + \sigma_t\right)\right]$$

we can obtain:

$$\sigma_z = E\varepsilon_z + 2\mu A = const.$$

As the axial strains ε_Z are constant (independent of radius) the same independency is confirmed also for axial stresses σ_Z.

For a cylinder with **inner radius r_1 and outer radius r_2 , loaded by pressure p_1 on its inner surface and by pressure p_2 on its outer surface**, the boundary conditions for calculation of integration constants can be formulated as follows:

for $r = r_1 => \sigma_r = -p_1$

for $r = r_2 => \sigma_r = -p_2$

By substituting the boundary conditions into eqs. (7a) and (7b), we can calculate the integration constants and obtain the resulting formulas for stresses:

$$\boxed{\begin{aligned}\sigma_r &= \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} - \frac{\left(p_1 - p_2\right) r_1^2 r_2^2}{r_2^2 - r_1^2}\frac{1}{r^2} \\ \sigma_t &= \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} + \frac{\left(p_1 - p_2\right) r_1^2 r_2^2}{r_2^2 - r_1^2}\frac{1}{r^2}\end{aligned}}$$

(8a)

(8b)

Procedure of the solution to a forward problem:

1. Evaluation of the **integration constants** in eqs. (7a) and (7b) on the basis of boundary conditions (known radial stresses).

2. Evaluation of the **axial stress** σ_z either from the equation of static equilibrium in ***z*** direction or (for a known axial strain) by using Hooke's law.

3. Analysis of the **stress distribution**, definition of dangerous points.
   The following **conclusions** can be drawn from the analysis of equations (7a), (7b):
   - Both stresses depend on radius - the dependence is polytropic.
   - The stresses are symmetric with respect to $A = (\sigma_t + \sigma_r)/2$
   - The difference between stresses σ_t, σ_r decreases with increasing radius.
   - Under any load gradients of σ_r and σ_t decrease in their absolute values with increasing radius.

4. Calculation of principal stresses in the dangerous point, evaluation of the **factor of safety** by using a reduced (equivalent) stress (based on some criterion of failure).

5. Calculation of **radial displacements** – the simplest way is on the basis of the circumferential strain, which can be calculated by using the Hooke's law.

Design of a cylindrical pressure vessel – inverse problem

(closed cylinder loaded by inner pressure)

Objective: proposal of the wall thickness (inverse problem) of a vessel with a given inner radius r_1, loaded by inner pressure p_1 (with a required factor of safety k_y).

It holds for the principal stresses: $\sigma_t \geq \sigma_z \geq \sigma_r$

For calculation of the equivalent (reduced) stress σ_{red} we can use Tresca's plasticity criterion

$$\sigma_{red} = \sigma_1 - \sigma_3 = \sigma_{t(r=r_1)} - \sigma_{r(r=r_1)} = A + \frac{B}{r_1^2} - A + \frac{B}{r_1^2} = \frac{2B}{r_1^2}$$

Integration constant B can be calculated from eq. (7a) for the boundary conditions of inner pressure p_1 and zero outer pressure:

$$B = p_1 \cdot \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}$$

By substituting for B in the above formula for equivalent stress and some mathematical manipulations we obtain the following equation for the unknown external radius r_2 of the vessel:

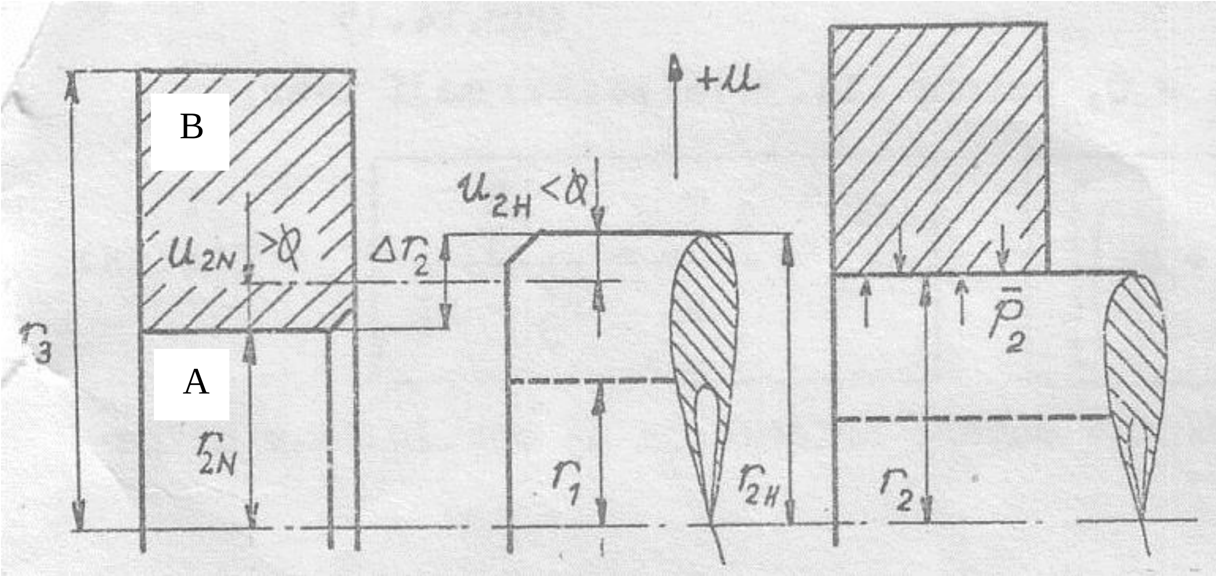
$$r_2 = r_1 \cdot \sqrt{\frac{\sigma_{red}}{\sigma_{red} - 2p_1}} \qquad (9)$$

In the limit state it holds $\sigma_{red} = \sigma_y = R_e$. A direct consequence of this formula is that no pressure vessel can bear a pressure higher than $p_1 = R_e/2$. Moreover, some factor of safety k_y is necessary in practical applications and the allowable stress σ_{all} ($\sigma_{all} = \frac{R_e}{k_y} = \sigma_{red}$) must not be exceeded; this value depends on the given material and the required factor of safety k_y.

This limitation can be overcome by means of:

- a better material with higher yield stress.
- autofrettage – inducing negative (compression) residual stresses at the inner surface of the vessel by exceeding its yield stress locally – done with liquid medium (water) to reduce the impact of failure. The procedure enables us to (nearly) double the load-bearing capacity of the vessel.
- a multilayer vessel with interference between adjacent layers.

Two-layer pressure vessel with interference



For a two-layer cylindrical pressure vessel loaded by pressure p_1 on the inner surface with radius r_1, with the outer radius r_3 and the interface radius $r_2 \approx r_{2N} \approx r_{2H}$, it holds from equations (8a) and (8b):

$$\sigma^A{}_r = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} - \frac{(p_1 - p_2) r_1^2 r_2^2}{r_2^2 - r_1^2} \frac{1}{r^2}$$

(10A)

$$\sigma^A{}_t = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} + \frac{(p_1 - p_2) r_1^2 r_2^2}{r_2^2 - r_1^2} \frac{1}{r^2}$$

for the inner body A

and

$$\sigma^B{}_r = \frac{p_2 r_2^2}{r_3^2 - r_2^2} - \frac{p_2 r_2^2 r_3^2}{r_3^2 - r_2^2} \frac{1}{r^2}$$

(10B)

$$\sigma^B{}_t = \frac{p_2 r_2^2}{r_3^2 - r_2^2} + \frac{p_2 r_2^2 r_3^2}{r_3^2 - r_2^2} \frac{1}{r^2}$$

for the outer body B

Here p_2 is the pressure acting at the interface because of the existing interference. The magnitude of the interference equals to the algebraic subtraction of radial displacements of both bodies in this location (displacements oriented outwards are positive in the applied sign convention):

$$\Delta r_2 = u_2^B - u_2^A \tag{11}$$

As **no centrifugal loads** are acting on the bodies in this case, the radial displacement of the inner body A is negative and the subtraction can be replaced by summation of absolute values of both displacements:

$$\Delta r_2 = \left| u_2^A \right| + \left| u_2^B \right|$$

If all the nominal dimensions are known, this formula enables us to calculate the magnitude of interference Δr_2 (or $\Delta d_2 = 2.\Delta r_2$ for diameter) needed to achieve the required pressure p_2 on the interface or to calculate the pressure p_2 for a given interference (in the assembly state, i.e. with $p_1 = 0$). We express the displacements from the circumferential strain and Hooke's law, and then substitute equations (10A) and (10B) for the stresses; thus we obtain:

$$u_2^A = \frac{r_2}{E}\left[\frac{-p_2 r_2^2}{r_2^2 - r_1^2} - \frac{p_2 r_1^2}{r_2^2 - r_1^2} + \mu\, p_2\right] \qquad u_2^B = \frac{r_2}{E}\left[\frac{p_2 r_2^2}{r_3^2 - r_2^2} + \frac{p_2 r_3^2}{r_3^2 - r_2^2} + \mu\, p_2\right]$$

Then we substitute the results into eq. (11) with pressure p_1 being zero and obtain the final formula for interference in the unloaded state (i. e. for assembly state and related to the diameter as prescribed in technical drawings):

$$\Delta d_2 = 2\Delta r_2 = \frac{4 p_2 r_2^3}{E} \frac{r_3^2 - r_1^2}{\left(r_3^2 - r_2^2\right)\left(r_2^2 - r_1^2\right)}$$

To set the optimal interface radius as well as optimal pressure at the interface, the following formulas can be used, derived on the basis of assumption of the same magnitude of maximum reduced stresses in both layers:

$$r_{2opt} = \sqrt{r_1 r_3} \quad \text{and} \quad p_{2opt} = \frac{p_1}{2}$$

The above calculations are valid exactly for vessels (layers) with the same lengths. If this is not the case (e.g. a wheel forced on a hollow shaft with a much higher length in z direction), stress (and contact pressure) concentrations occur at the edges of the contact surface. The extreme stresses may exceed even the yield stress in some cases but they are not dangerous because the plastic deformation is very local (in a very small volume) and cannot be repeated to induce fatigue failure.

Full shaft with a wheel

For a full shaft the boundary conditions are different. To keep the same notation and subscripts, let's assume that a shaft with radius r_2 is loaded by pressure p_2 on its outer surface (contact pressure from a wheel forced on it); then the boundary conditions for calculation of integration constants can be formulated as follows:

for $r = 0 \Rightarrow \sigma_r = \sigma_t$

for $r = r_2 \Rightarrow \sigma_r = -p_2$

By substituting the BCs into eqs. (7a) and (7b), we can calculate the integration constants ***B* = *0*** and ***A*= – *p*2** and obtain the resulting formula for stresses

$$\sigma_r = \sigma_t = const = -p_2$$

Both stresses are independent of radius and equal.