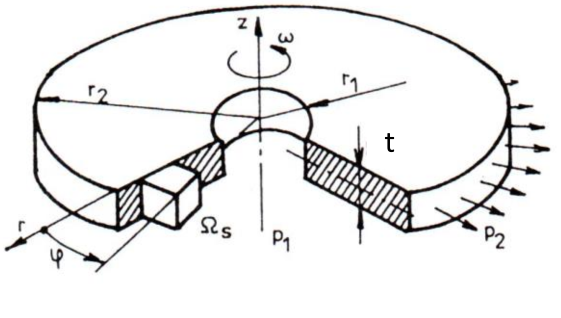
evropský sociální fond v ČR
EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Rotating disc (axisymmetric wall)

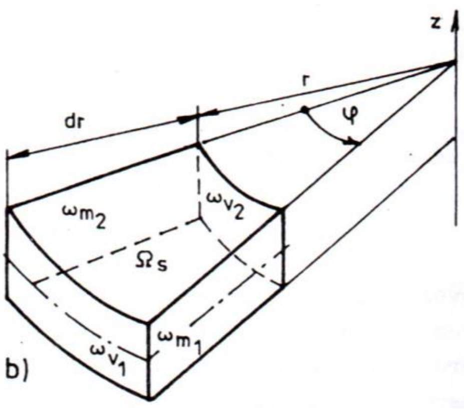
It is a thin-wall body with a planar (circular or annular) middle surface, keeping its planarity also in the deformed state. The load is allowed to act in the middle plane only; this condition together with the axisymmetry can be met only by centrifugal forces larger by order than gravitational forces, which can be neglected in this case. Practical applications represent high speed rotating disc with circular or annular middle plane, like circular saws, rotors of turbines and other engines, etc.



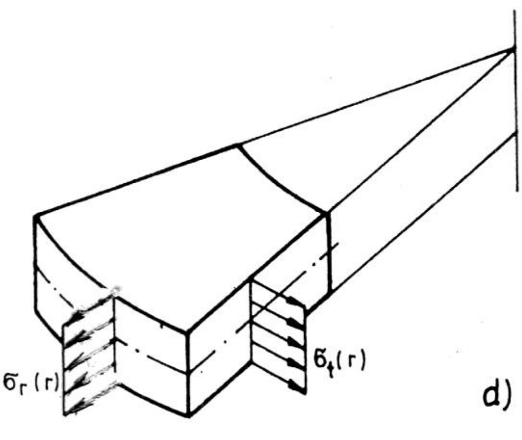
a)

The basic geometrical difference from the cylindrical thick-wall body is that the disc (wall) is thin ($t << r_2$), which results in the following simplifications:

- A twice infinitesimal element is sufficient for creation of a free body diagram (see fig. b)
- The state of stress is only two-dimensional (planar - see fig. d) and constant throughout the thickness.



b)

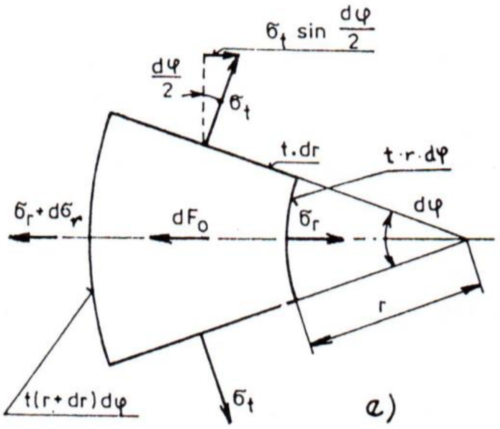


d)

1. **Strain-displacement equations** (in the cylindrical coordinate system) are identical with a cylindric body:

$$\varepsilon_r = \frac{du}{dr}\,; \quad \varepsilon_t = \frac{u}{r}\,; \quad \varepsilon_z = \frac{dw}{dz}$$

2. Formulation of **equations of static equilibrium** – we apply only one force equation for the radial direction, it can be manipulated to obtain the following form:



$$\sigma_r - \sigma_t + r\frac{d\sigma_r}{dr} = -\rho r^2\omega^2 \qquad (1)$$

It holds $\sigma_z = 0$ for the axial stress, because the stress state is planar here.

3. **Hooke's law** will be applied (with explicitly expressed stresses) in its specific form valid for 2D stress state:

$$\sigma_r = \frac{E}{1-\mu^2}\left(\varepsilon_r + \mu\varepsilon_t\right);\ \sigma_t = \frac{E}{1-\mu^2}\left(\varepsilon_t + \mu\varepsilon_r\right)$$

By differentiation of the first equation with respect to r and substitution of the strain-displacement equations, the following equation is obtained:

$$\frac{d\sigma_r}{dr} = \frac{E}{1-\mu^2}\left[\frac{d^2u}{dr^2} + \mu\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right)\right] \qquad (2)$$

We subtract both equations of the Hooke's law to obtain

$$\sigma_r - \sigma_t = \frac{E}{1+\mu}\left(\varepsilon_r - \varepsilon_t\right)\ , \qquad (3)$$

and by substituting the strain-displacement equations we can obtain

$$\sigma_r - \sigma_t = \frac{E}{1+\mu}\left(\frac{du}{dr} - \frac{u}{r}\right) \qquad (4)$$

Then we substitute eqs. (2) and (4) into eq. (1) and after some manipulations we obtain the following differential equation for radial displacements:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -\frac{1-\mu^2}{E}\rho r\omega^2 \qquad (5)$$

Eq. (5) differs from the analogical equation valid for the cylindrical vessel by its right-hand side only – it is a non-homogeneous differential equation.

$$u = u_{\mathrm{hom}} + u_{part} \tag{6}$$

The particular integral of the non-homogeneous differential equation can be found in the following form:

$$u_{part} = -\frac{1-\mu^2}{8E}\rho\omega^2 r^3 \tag{7}$$

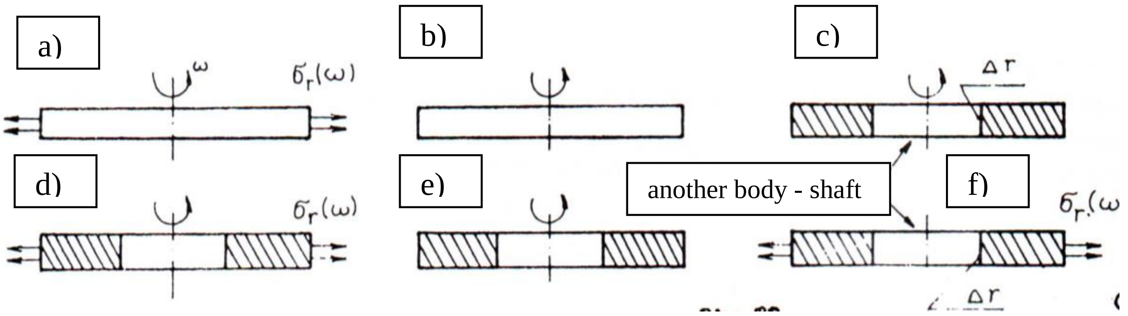
The resulting equation for radial displacements has the shape

$$u = c_1 r + \frac{c_2}{r} - \frac{1-\mu^2}{8E}\rho\omega^2 r^3 \tag{8}$$

and enables us to determine strains by using the strain-displacement equations; by substituting the results into the Hooke's law and introducing new forms of the integration constants, we can obtain final formulas for stress components in the following form:

$$\boxed{\begin{aligned} \sigma_r &= A - \frac{B}{r^2} - \frac{3+\mu}{8}\rho\omega^2 r^2 \qquad \text{(9a)} \\ \sigma_t &= A + \frac{B}{r^2} - \frac{1+3\mu}{8}\rho\omega^2 r^2 \qquad \text{(9b)} \\ \sigma_z &= 0 \qquad \text{(9c)} \end{aligned}}$$

Similarly to a cylindric body, coordinate axes *r, t, z* correspond to the principal directions; formulation of boundary conditions for calculation of the integration constants is (analogically to the cylindrical body) based on the loads acting onto the inner and outer surfaces. Most typical situations are as follows:



The most important formulations of boundary conditions:

1) **Free (unloaded) rotating annulus** – rotating wall with a central hole

Boundary conditions:

for $r = r_1 => \sigma_r = 0$

for $r = r_2 => \sigma_r = 0$

By substituting the BCs into eqs. (9a) and (9b), we can calculate the integration constants and obtain the resulting formulas for stresses

$$\boxed{\begin{aligned}\sigma_r &= \frac{3+\mu}{8}\rho\omega^2\left(r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2\right) \\ \sigma_t &= \frac{3+\mu}{8}\rho\omega^2\left(r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{1+3\mu}{3+\mu}r^2\right)\end{aligned}}$$

(10a)

(10b)

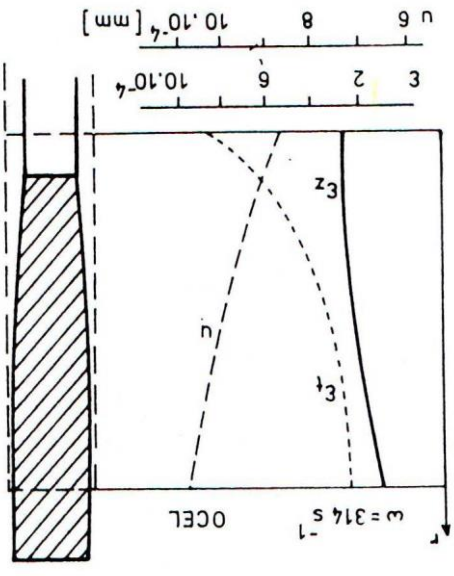
We can calculate also the radial displacement of the outer surface (for $r = r_2$)

$$u = \varepsilon_t . r_2 = \frac{r_2}{E}\sigma_{t(r=r_2)} = \frac{3+\mu}{8E}\rho\omega^2\left[\left(r_1^2 + r_2^2\right)r_2 + r_1^2 r_2 - \frac{1+3\mu}{3+\mu}r_2^3\right]$$

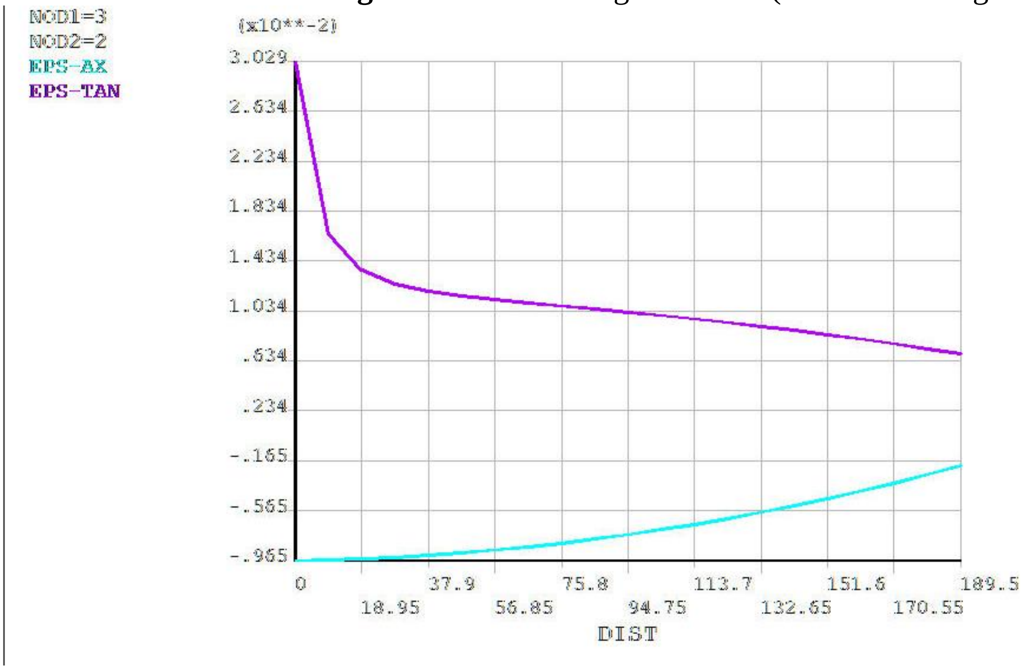
Also the axial strain can be calculated using Hooke's law

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_t + \sigma_r\right)\right] = -\frac{\mu}{E}\left[2A - \frac{1}{8}\rho\omega^2 r^2\left(4+4\mu\right)\right] = \frac{\mu}{E}\left[\frac{1+\mu}{2}\rho\omega^2 r^2 - 2A\right]$$

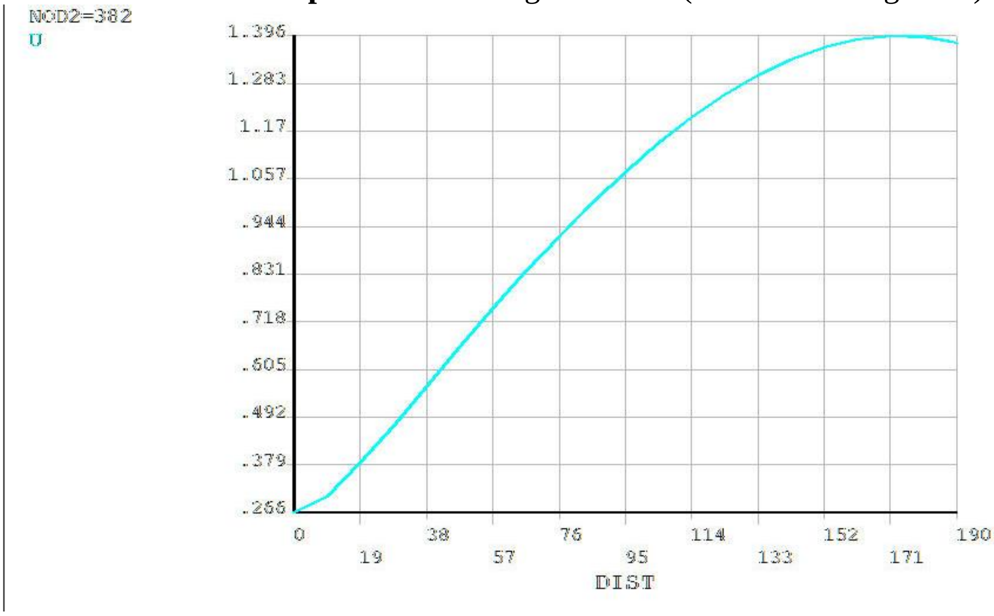
It is obvious that the change of the wall thickness (ε_z<0) is not constant, its dependence on radius is parabolic. Distribution of displacement u > 0 is more complex, given by eq. (8), while distribution of ε_t = u/r > 0 is more progressively increasing with decreasing radius (see the figure below, more realistic distributions calculated using FEM are depicted at the next page).



Distribution of **axial and tangential strains** along the radius (calculated using FEM)



Distribution of **radial displacements** along the radius (calculated using FEM)



2) Free rotating hollow shaft

Axial deformation is constrained, which results in non-zero axial stresses as a consequence of transversal contraction; these stresses are positive near the inner surface and negative near the outer surface. Consequently, the axial stresses at the dangerous location (the inner surface) are positive, it holds $\sigma_z = \sigma_2 > 0$ ($\sigma_t \geq \sigma_z \geq \sigma_r$) and these stresses do not influence the reduced stress calculated by using Tresca's criterion.

3) Free rotating disc (without a hole)

Boundary conditions:

for $r = 0 => \sigma_r = \sigma_t$

for $r = r_2 => \sigma_r = 0$

It follows from these BCs after their substitution into eqs. (9a) and (9b) that the constant B=0. Then it holds for stress components:

$$\boxed{\begin{aligned} \sigma_r &= \frac{3+\mu}{8}\rho\omega^2\left(r_2^2 - r^2\right) \\ \sigma_t &= \frac{3+\mu}{8}\rho\omega^2\left(r_2^2 - \frac{1+3\mu}{3+\mu}r^2\right) \end{aligned}}$$

The stress distributions are parabolic in this case with maximum value being in the centre of the disc.

4) Free rotating shaft (without a hole)

Axial deformation is constrained, which results in non-zero axial stresses as a consequence of transversal contraction; these stresses are positive near the centreline of the shaft and negative near its surface. Consequently, the axial stresses at the dangerous location (centre of the cross section) are positive, it holds here $\sigma_z = \sigma_3 > 0$, and these stresses reduce slightly the reduced stress calculated on the basis of Tresca's criterion. Consequently, application of the above theory is safe, the real factor of safety is higher than the calculated one.

5) Rotating disc with a compressive (tensional) load on the outer surface

The stress distribution is parabolic similarly to case 3) but the parabolas are shifted up (for tensional load) or down (for compressive load) by the value of pressure acting on the outer surface of the disc. Consequently, if the load is compressive, the position of the dangerous point is uncertain (it can be either in the centre or on the outer surface, depending on the magnitudes of the rotation speed and pressure).

6) Free rotating disc with a small hole

The eqs. (10) derived for case 1) are valid here as well but they can be simplified with respect to a negligible value of the r_1 radius:

$\sigma_r = 0$; $r_1 \rightarrow 0$

$$\sigma_{t(r=r_1)} = \sigma_{red} = \frac{3+\mu}{8}\rho\omega^2\left(r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r_1^2} - \frac{1+3\mu}{3+\mu}r_1^2\right) =$$

$$= \frac{3+\mu}{8}\rho\omega^2\left(r_2^2 + \frac{r_1^2 r_2^2}{r_1^2}\right) = \frac{3+\mu}{4}\rho\omega^2 r_2^2$$

The reduced stress is two times higher than for a disc without the hole (radial or tangential stress for r = 0 in case 3)).

Conclusion:

A small hole induces a stress concentration in its surroundings with the **stress concentration factor of 2**.

7) Thin rotating ring

The following simplification holds here: $r_1 \approx r_2 = R$. Then eq. (10a) yields the result $\sigma_{r1} = \sigma_{r2} = 0 = \sigma_r$ and it holds from the eq. (10b):

$$\sigma_t = \frac{3+\mu}{8}\rho\omega^2\left(r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{1+3\mu}{3+\mu}r^2\right) =$$

$$= \frac{3+\mu}{8}\rho\omega^2\left(R^2 + R^2 + \frac{R^2 R^2}{R^2} - \frac{1+3\mu}{3+\mu}R^2\right) = \frac{1}{8}\rho\omega^2 R^2(9+3\mu-1-3\mu) = \rho\omega^2 R^2$$

Due to a negligible thickness of the ring it can be assumed that the stress is constant throughout all the volume of the ring, similarly to simple tension of bars.