Axisymmetric plate

is a thin-wall body with a planar (not curved) middle surface in its undeformed state; it can be a circle or annulus. The **load** acts in the direction **perpendicular to the middle plane**, so that the plate is bended and its middle plane becomes an axisymmetric curved (skewed) surface after deformation. **Deflection w** (displacement in the axial direction) is the major parameter of deformation; **slope υ** (tangenta of the rotation angle) is introduced as a slave parameter of deformation.

Applications:
end-plates of vessels, tops of pistons, pressure sensors, etc.

Stress tensor corresponds to a general axisymmetric body, with one of the principal stresses (σ_z) being zero because of the tiny plate dimension in the axial direction (thickness); the matrix form of the tensor can be written as follows:

$$T_\sigma = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_t & 0 \\ \tau_{rz} & 0 & 0 \end{bmatrix}$$

Systemization of the plates:
With respect to their relative thickness, the plates can be divided into several groups:

a) **Thick plates**

After deformation the deflections are very small, elongations of radial fibres are negligible, as well as the membrane stresses. Bending (normal) stresses are on the same order as shear stresses (similarly to thick beams) and both **flexion and shear loads** should be taken into account. Normal lines of the middle surface do not only rotate but become curved as well. The **Mindlin** theory is based on the simplifying assumption that these normal lines remain straight but not perpendicular to the middle surface. Not very frequent in technical applications.

b) **Thin plates with small deflections**

The common thickness limit for thin plates is **$h<R/10$**. Then shear stresses are irrelevant from the point of view of failure which is dominated by normal stresses. In addition however, the deflection must be as small that the problem remains linear in geometry (the common limit is **$w<h/4$**). The **Kirchhoff theory of plates** assumes that the normal lines of the middle surface remain straight and perpendicular to it; the elongation of radial fibres is **negligible,** as well as the **membrane stresses**. Only **bending (normal) stresses** are taken into account; they are distributed linearly throughout the plate thickness with zero value in the middle surface. This theory offers the simplest calculations among all theories of plates and is **the most frequent in technical applications**.

c) **Thin plates with large deflections**

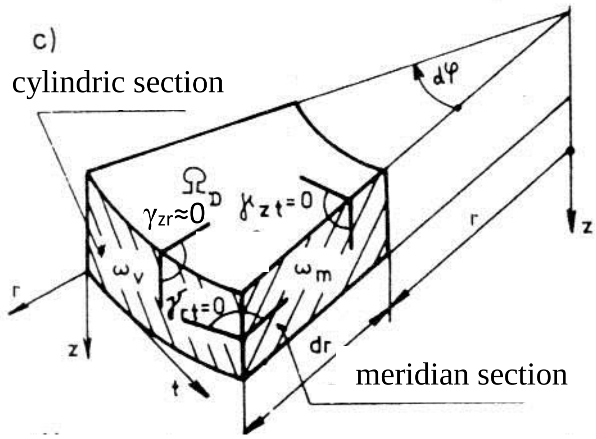
The flexural rigidity (related to thickness) of these plates is lower than of those in par. b). Thickness limits cannot be defined exactly, because they depend on the material parameters and load magnitude; the limits are given by the deflection magnitude ($h/4<w<5h$). Large deformations of these plates require a non-linear solution (non-linear geometrical relations under load) and the membrane stresses need to be taken into consideration as well.

d) **Membranes**

They are as thin that their flexural rigidity is negligible, their calculations take only tensional load into account (normal stresses uniform throughout the thickness, membrane stress state); the membrane theory of shells (chapter 10) can be used only under condition their large displacements (deformed geometry) are taken into account. Also here their relative deflection under load, rather than their thickness, is the conventional limiting quantity ($w>5h$).

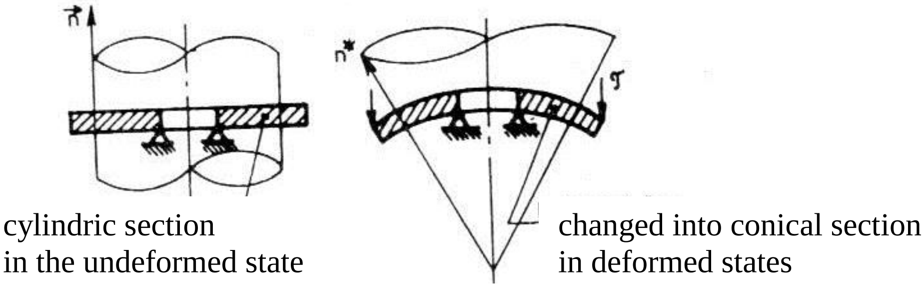
Kirchhoff theory of thin axisymmetric plates

A typical element is twice infinitesimal:



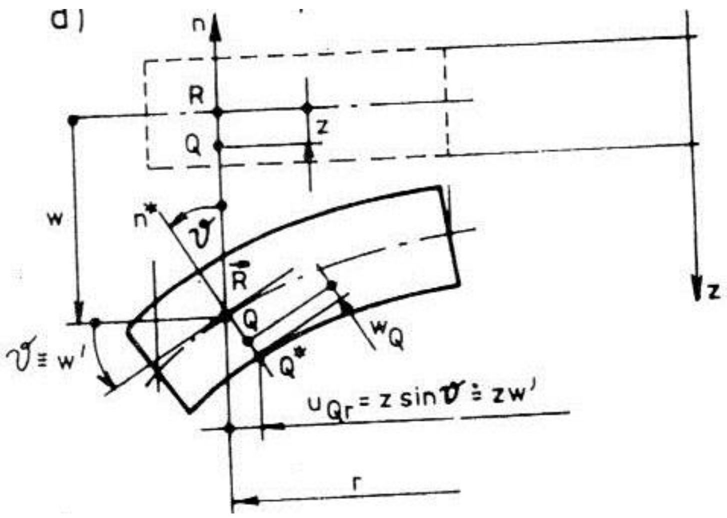
Basic assumptions

1. Stress and strain states are axisymmetric –tangential direction is a principal direction (see the figure above).
2. Normal lines of the middle surface remain straight and perpendicular to this surface (see the figure below), consequently cylidrical sections change into conical ones, and the strains and stresses are distributed linearly throughout the plate thickness.



3. $\tau_{rz} \approx 0$, shear stress is negligible from the point of view of failure; however, this stress is necessary to equilibrate the element of the plate.
4. The stresses perpendicular to the middle surface (σ_z) are negligible (because of small thickness of the plate).
5. Points in the middle plane show negligible radial displacements ($u_R = 0$), consequently membrane stresses are negligible.

Relations between deformation parameters

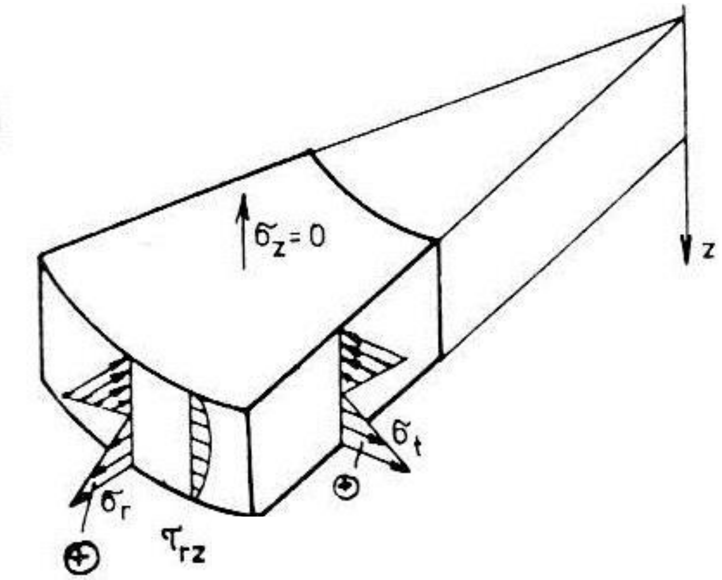


$$\upsilon = tg\,\upsilon = \frac{dw}{dr} = w' \qquad (1)$$

$$u = -z\upsilon \qquad (2)$$

Minus is added in eq. (2) due to the orientation of *z* axis; for positive *z* coordinate we obtain negative radial displacement *u* and vice versa.

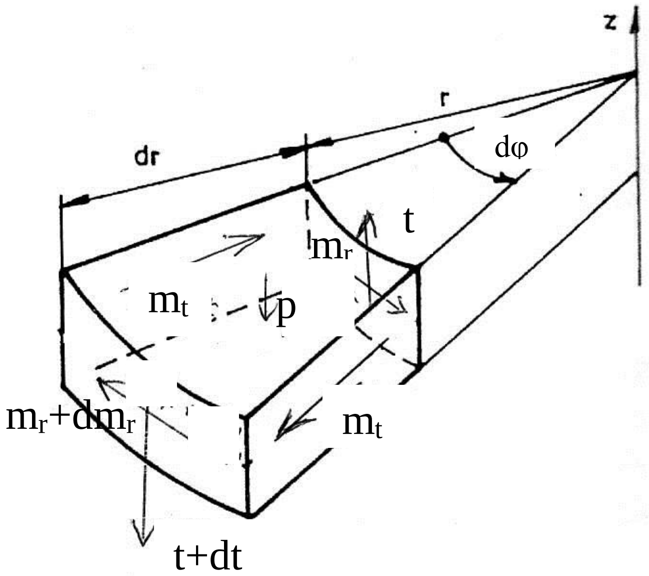
Stresses in an infinitesimal element



The stresses can be replaced by their **resulting loads distributed along lines (forces** and **couples per unit length** – see figure below**)** on the basis of the following equations of **static equivalence**:

$$t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{rz} dz \; [N/mm] \tag{3a}$$

$$m_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_r dz \qquad m_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_t dz \; [Nmm/mm] \tag{3b}$$



System of equations used in the solution

Equations of static equilibrium:

$$\sum F_z = 0: \quad \frac{dt}{dr} + \frac{t}{r} + p(r) = 0 \tag{4a}$$

$$\sum M_t = 0: \quad m_r - m_t + r\frac{dm_r}{dr} = rt \tag{4b}$$

Strain-displacement equations:

$$\varepsilon_r = \frac{du}{dr} = -z\frac{d\upsilon}{dr} \tag{5a}$$

$$\varepsilon_t = \frac{u}{r} = -z\frac{\upsilon}{r} \tag{5b}$$

Constitutive equations (for plane stress conditions):

$$\sigma_r = \frac{E}{1-\mu^2}\left[\varepsilon_r + \mu\varepsilon_t\right] \tag{6a}$$

$$\sigma_t = \frac{E}{1-\mu^2}\left[\varepsilon_t + \mu\varepsilon_r\right] \tag{6b}$$

The procedure of solution:
Strain-displacement equations (5) are substituted into the constitutive equations (6), and the results further into the equations of static equivalence (3b). After some manipulations we can obtain:

$$m_r = -\frac{Eh^3}{12\left(1-\mu^2\right)}\left[\frac{d\upsilon}{dr}+\mu\frac{\upsilon}{r}\right] = -B\left[\frac{d\upsilon}{dr}+\mu\frac{\upsilon}{r}\right] \tag{7a}$$

$$m_t = -\frac{Eh^3}{12\left(1-\mu^2\right)}\left[\frac{\upsilon}{r}+\mu\frac{d\upsilon}{dr}\right] = -B\left[\frac{\upsilon}{r}+\mu\frac{d\upsilon}{dr}\right] \tag{7b}$$

In these equations the multiplicand in front of the brackets represents the **flexural rigidity B** of the plate

$$B = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{8}$$

The formulas (7) and a derivative of the formula (7a) can be substituted into the momentum equation of static equilibrium (4b), and after some algebra we can obtain the following differential equation (containing the slope υ as the unknown function):

$$\frac{d^2\upsilon}{dr^2}+\frac{1}{r}\frac{d\upsilon}{dr}-\frac{1}{r^2}\upsilon = -\frac{t(r)}{B} \tag{9}$$

A general solution to eq. (9) exists in the following form:

$$\upsilon(r) = c_1 r + \frac{c_2}{r} + \upsilon_p \tag{10}$$

Here υ_p represents a particular integral of the non-homogeneous differential equation (9); its form depends on the form of the function t(r) (shear force distributed along a line).

Eq. (9) can be transformed into the following form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\upsilon\right)\right] = -\frac{t(r)}{B} \tag{11}$$

which makes the solution possible by two successive integrations.

On the basis of eq. (1), another integration of eq. (10) gives the relation for the deflection *w* of the plate:

$$w(r) = c_1\frac{r^2}{2} + c_2\ln r + w_p + c_3 \tag{12}$$

Boundary conditions

can be formulated on the basis of constraints of deformation parameters (supports), external loads (couples distributed along line), axisymmetry of the plate middle surface (for a circular plate without a central hole), or of continuity and smoothness of the middle surface (at the boundaries between individual intervals).

- Supports:
  - Fixed support: w=0, υ=0=w' .
  - Pin or roller support (both are equivalent because of negligible forces acting in the middle plane): w=0.
- Free edge (not fixed and unloaded): m_r=0
- When a distributed line couple acts on the free edge, the radial moment here is not zero but equals to the magnitude of this couple.
- A plate without any hole: for r=0 it holds υ=0 => c_2=0.
- It is necessary to divide the plate into intervals in those locations (with exception of the plate edges, naturally) where it is
  - discontinuity in loads (in the form of a support, concentrated line load, change in the character of distributed loads),
  - change of the plate thickness or in material parameters.

  Each of these intervals requires a specific differential equation coupled with the equations for the neighbouring intervals by three boundary conditions at each boundary. These boundary conditions are based on the equality of deflections, slopes and radial moments at the boundary.
- If an external distributed line couple acts on a circular line defined by its radius (except for the plate edges), a stepwise change occurs in the values of the radial moment; the magnitude of this step equals to the magnitude of the distributed line couple acting here.

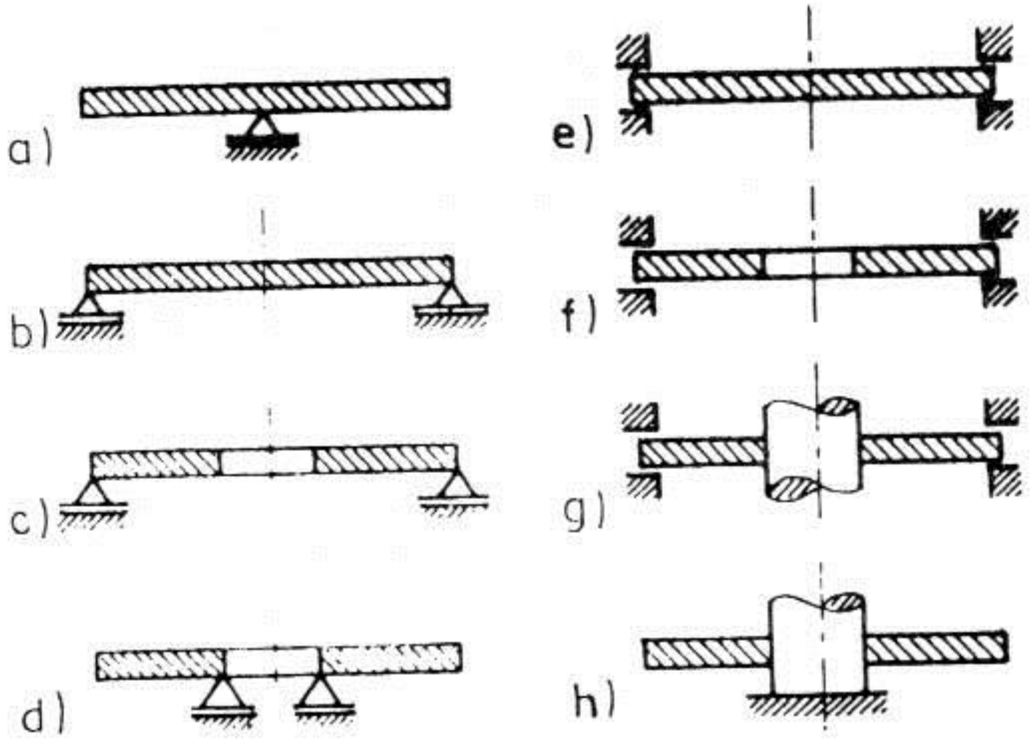
Evaluation of stresses

When the boundary conditions are solved and the integration constants known, we are able to calculate stresses. By substitution of eq. (10) into eqs. (7) we obtain formulas for both bending moments. Then we apply eqs. (3b) and the assumption of linear distribution of stresses (σ_r= $A.z$) which gives for radial moment m_r (and analogically for tangential moment m_t):

$$m_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_r dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} Az^2 dz = \left[\frac{Az^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = A\frac{h^3}{12} \quad => \quad A = \frac{12m_r}{h^3} \quad => \quad \sigma_r = \frac{12m_r}{h^3} z$$

Maximum stress on the surface $(z = \pm h/2)$ is then $\sigma_{r\max} = \pm 6m_r / h^2$ and $\sigma_{t\max} = \pm 6m_t / h^2$.

Typical examples of supports of axisymmetric plates



Procedure of solution to a direct problem:

Note: *The force equation of static equilibrium (4a) was not used in the solution. The static equilibrium of z components of forces is then used in evaluation of the distributed line shear force; however, it is easier to formulate this equation in another form, namely for a finite element of the plate separated by a cylindrical section.*

1. The plate is divided into intervals along its radius; in each of them the shear force must be defined by a single continuous and smooth function, and the plate thickness and modulus of elasticity must be constant.

2. For each of the intervals, a finite element (cut off by a cylindrical section with variable radius r) is isolated from the plate as a free body, and the distributed line shear force $t(r)$ is calculated from the z-axis equation of static equilibrium.

3. The slope $\upsilon(r)$ can be determined by two consecutive integrations of eq. (11) or by substituting the particular integral υ_p into eq. (10); deflection $w(r)$ can then be obtained by another integration of the $\upsilon(r)$ function.

4. We formulate boundary conditions (3 for each of the intervals) and calculate the unknown integration constants in the obtained equations.

5. By substituting $\upsilon(r)$ into eqs. (7), we can calculate the distributed line moments as functions of the radius r and draw their dependences on the radius.

6. We find the dangerous points of the plate, i.e. extremes (maxima) of the moments expressed as functions of the radius. Extreme stresses in the dangerous points can be calculated from the following formulas:

$$\sigma_{r\max} = \pm\frac{6m_r}{h^2} \quad \text{a} \quad \sigma_{t\max} = \pm\frac{6m_t}{h^2}$$

7. As the third principal stress σ_z equals zero, the reduced Tresca stress (valid for a ductile material) equals to the magnitude of the larger one of both of the above stresses. This value is then used for calculation of the factor of safety.

8. The deflection function $w(r)$ can be obtained by substituting the integration constants into eq. (12) (together with the known function w_p). For validity of the solution, the maximum of this function must meet the condition

$$w_{max} < h/4$$