11. Simple tension and compression

11.1. Definition

Simple tension (compression) is a type of loading of a straight prismatic bar, if

- bar assumptions are satisfied,
- cross sections mutually draw away (near) and consequently deform isotropically (i.e. they change their magnitude but not the shape),
- normal force $\vec{N}$ is the only one non-zero component of the inner resultants,
- deformations are not substantial from the viewpoint of static equilibrium of an element.

bar assumptions

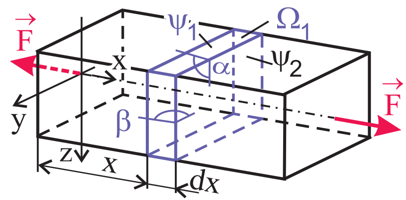
Relations for stresses and deformations will be derived on the basis of these assumptions.

Note: A prismatic bar has all the cross sections equal and the principal central axes of these sections do not rotate along the bar centreline (non-screw-shaped).

11.2. Geometrical relations

These relations describe the dependencies between displacements and strains. First, we express the length and angular strains in dependence on the type of the change of the mutual positions of cross sections during the loading process.

Under the tensile (compressive) load, the distance between the cross sections ψ_1 and ψ_2 of a onefold **infinitesimal** element Ω_1 being originally $\mathrm{d}x$ becomes **longer (shorter)** by the value of du; the (deformation) displacement $\mathrm{d}u$ is constant through all the points of the cross section. The **right angles** α and β do not change.

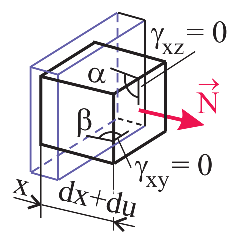


element

The following components of the strain tensor correspond to these deformations: (*Note:* we do not express all the components of the strain tensor but only those having at least one index x. According to the convention introduced above, x denotes the normal line of the cross section, so that ε_x, γ_{xy} and γ_{xz} define the position of the cross section and the corresponding components of stresses can be calculated using Hooke's law. All the other components of T_σ equal zero according to the bar assumptions concerning the stress state. A similar situation occurs at the other types of loading of bars.)

$\varepsilon_x = \dfrac{\mathrm{d}u}{\mathrm{d}x}, \gamma_{xy} = \gamma_{xz} = 0$
(the cross sections remain perpendicular to the bar centreline)
As the displacement $\mathrm{d}u$ is the same for all the points of the cross section $((\mathrm{d}u(y, z) = \text{konst.}))$,

$$\varepsilon_x(y, z) = \frac{\mathrm{d}u}{\mathrm{d}x} = \text{konst.}$$



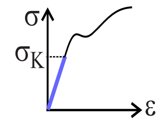
Therefore the strains ε_x are also constant through all the cross section.

The same statement holds for the transversal length strains ε_y and ε_z, which are also non-zero but have an opposite sign ($\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x$, where μ is Poisson's ratio).
Thus a **triaxial** state of stress comes into existence in the bar.

Strain tensor $T_\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$. strain tensor

11.3. Stress distribution in the cross section

Hooke's law



For a Hookean material (homogeneous, linear elastic), the following linear dependency holds

$$\sigma_x(y,z) = E\varepsilon_x(y,z).$$

As it holds $\varepsilon_x(y,z) = \text{const.}$, it holds also $\sigma_x(y,z) = \text{const.} = \sigma$ (distributed uniformly throughout the cross section ψ). $\sigma_y = \sigma_z = 0$ results from the bar assumptions concerning stress states (there is a bar-type stress state in ψ). bar assumptions

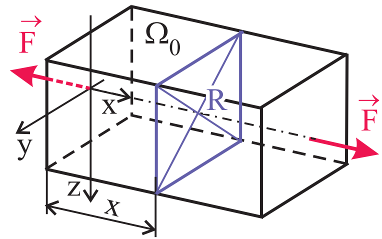
Shear stress is given by the constitutive relation $\tau_{ij} = \dfrac{E}{2(1+\mu)}\gamma_{ij} = G\gamma_{ij}$,
(G is a material characteristic dependent on E and μ for an isotropic material). G

As $\gamma_{xy} = \gamma_{xz} = 0$, it holds also $\tau_{xy} = \tau_{xz} = 0$. $\tau_{yz} = 0$ results from the bar assumptions.

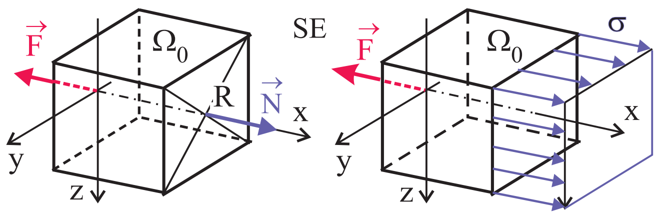
There is a **uniaxial** state of stress in the bar. Stress tensor $T_\sigma = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. T_σ

11.4. Relation between component of inner resultant and stress

As we know the stress distribution throughout the cross section, the continuous inner forces do not more represent an infinity of unknown parameters (for $\sigma =$ const. even one single parameter) and it is possible to express the dependency of the normal stress σ on the just one non-zero component of inner resultants. For this aim we use conditions of static equivalence between the system of elementary forces $\mathrm{d}N\vec{i} = \sigma_x \mathrm{d}S\,\vec{i}$ in the cross section and their force resultant $N\vec{i}$ acting in the gravitational center of the cross section.



We formulate the applicable conditions of static equivalence (a 3D system of parallel forces $\Rightarrow \quad \nu = 3$):



static conditions

$$\sum_{\psi} F_x : \quad \iint_{\psi} \sigma_x \mathrm{d}S = N,$$

$$\sigma_x = \text{konst.} \quad \Rightarrow \quad \iint_{\psi} \sigma_x \mathrm{d}S = \sigma_x \iint_{\psi} \mathrm{d}S \; = N \quad \Rightarrow \quad \boxed{\sigma = \frac{N}{S}}$$

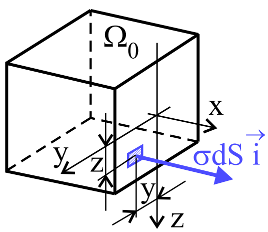
In the final formula we usually omit the subscript x at the stress symbol already, because all the others stress components equal zero. However, we derived this formula from the equation of static equivalence, which holds if and only if all the applicable conditions of

static equivalence

the static equivalence are satisfied. To ensure the applicability of this formula, we must check the validity of the remaining two conditions of static equivalence.

static moments

$$\sum_{\psi} M_y : \iint_{\psi} z\sigma_x \mathrm{d}S = M_{oy}, \quad \sum_{\psi} M_z : -\iint_{\psi} y\sigma \mathrm{d}S = M_{oz}.$$



It results from the definition of the simple tension that $M_{oy} = M_{oz} = 0$. Then both conditions can be manipulated to get the form

$$\iint_{\psi} z\,\sigma \mathrm{d}S = \sigma \iint_{\psi} z\,\mathrm{d}S = \sigma U_y = 0, \quad \iint_{\psi} y\sigma \mathrm{d}S = \sigma \iint_{\psi} y \mathrm{d}S = \sigma U_z = 0.$$

Both conditions of static equivalence are satisfied, because the coordinate axes y and z pass the **centroid** ($U_y = 0$, $U_z = 0$).

11.5. Extreme stress

limit states

For evaluation of the risk of limit states (failures) it is important to know the locations and magnitudes of extreme stresses in the cross sections. As it was derived above, the stress distribution is uniform throughout the cross section in the case of simple tension (compression), so that all points of the cross section have the same **factor of safety** and the extreme stress is given directly by the formula derived above

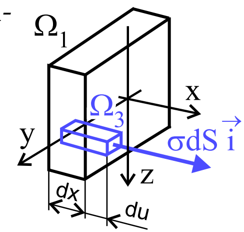
$$\sigma_{ex} = \frac{N}{S}\,.$$

11.6. Strain energy

In linear elastic bodies, all the deformation work takes effect in increase of the reversible elastic strain energy $A = \Delta W$ (the work done for permanent deformation equals zero $A_Q = 0$).

deformation work

There is an inner elementary force $\sigma \mathrm{d}S\vec{i}$ acting on the threefold infinitesi--mal element Ω_3. The change of the length $\mathrm{d}x$ of this element equals the increment denoted as $\mathrm{d}u$. The deformation work of the inner elementary force (we suppose a linear elastic body) $A_{\sigma \mathrm{d}S} = \frac{1}{2}(\sigma \mathrm{d}S)\mathrm{d}u$. After substitution $\mathrm{d}u = \varepsilon \mathrm{d}x$ and $\varepsilon = \sigma/E$ we obtain the following relation for the **strain energy** of the element in question



$$W_{\Omega_3} = A_{(\sigma \mathrm{d}S)} = \frac{1}{2}(\sigma \mathrm{d}S)\varepsilon \mathrm{d}x = \frac{1}{2}\frac{\sigma^2}{E}\mathrm{d}S\mathrm{d}x.$$

The strain energy related to a volume unit gives the so called strain energy density

$$\Lambda = \frac{W_{\Omega_3}}{V_{\Omega_3}} = \frac{W_{\Omega_3}}{\mathrm{d}S\mathrm{d}x} \qquad \Rightarrow \qquad \boxed{\Lambda = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\frac{\sigma^2}{E}}$$

These relations hold generally for a bar showing a uniaxial stress state defined by the stress σ independent of the type of loading of the bar.

For the simple tension it holds $\sigma = \frac{N}{S}$ and the strain energy of a onefold infinitesimal element Ω_1 is then given by the relation

$$W_{\Omega_1} = \iint_{\psi} \frac{1}{2}\frac{\sigma^2}{E}\mathrm{d}x\mathrm{d}S = \iint_{\psi} \frac{N^2}{2ES^2}\mathrm{d}x\mathrm{d}S = \frac{N^2\mathrm{d}x}{2ES^2}\iint \mathrm{d}S = \frac{N^2}{2ES}\mathrm{d}x.$$

The total strain energy accumulated in the bar of length l is

$$\boxed{W_l = \int_0^l W_{\Omega_1} = \int_0^l \frac{N^2}{2ES}\mathrm{d}x\,.}$$

11.7. Deformation characteristics of the centreline

The basic deformation characteristic of a beam loaded by simple tension is a displacement of a point of the centreline in the direction parallel with the centreline. The length strain of the centreline $\varepsilon_x = \mathrm{d}u/\mathrm{d}x$. As the bar assumptions are satisfied, the centreline remains continuous and the displacements $u(x)$ represent a continuous function. For a Hookean material $\left(\varepsilon_x = \frac{\sigma_x}{E}\right)$ and for loading by the simple tension $\left(\sigma_x(x_R) = \dfrac{N(x_R)}{S(x_R)}\right)$, the displacement of the point R defined by the coordinate x_R equals

bar assumptions

Hookean material

$\sigma_x(x)$

$$u(x_R) = \int\limits_{x_m}^{x_R} \varepsilon_x \mathrm{d}x = \int\limits_{x_m}^{x_R} \frac{N(x)}{ES(x)} \mathrm{d}x,$$

where x_m is the coordinate of the point of the centreline having zero displacement (usually a point of support - fixed to the base.).

Example 432

Example 433

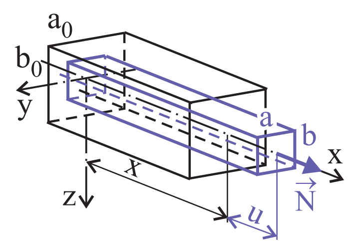
The modulus of elasticity E can theoretically change along the centreline (in the cross section, however, it must be constant to satisfy the assumptions of simple tension), but in practice you meet only a **stepwise** change (different materials **along** the length of the bar).

assumptions

If it holds $N(x) = \text{const.}$, $E(x) = \text{const.}$ and $S(x) = \text{const.}$ in some interval of the bar centreline, and if the point of zero displacement is identical with the origin of the coordinate system ($x_m = 0$), then

$$u(x_R) = \frac{N x_R}{ES},$$ where ES is called the **stiffness of the cross section in tension**.

11.8. Deformations of the cross section



Besides of the longitudinal displacements of the cross section, also a change in its dimensions occurs under conditions of simple tension (compression).
Poisson's ratio μ characterises the ratio between the transversal strain (ε_y or ε_z) and the longitudinal strain (ε_x), therefore

$$\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x$$

As the strains in both transversal directions are equal and constant throughout the cross section, the shape of the cross section does not change. The definition equations of the strains being constant throughout the rectangular cross section can be solved for changes in cross section dimensions Δa and Δb:

$$a - a_0 = \Delta a = \varepsilon_y a_0 = -\mu\varepsilon_x a_0,$$

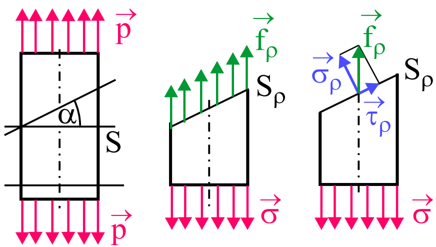
$$b - b_0 = \Delta b = \varepsilon_z b_0 = -\mu\varepsilon_x b_0.$$

11.9. Analysis of the state of stress under simple tension

Till now, we evaluated stresses in a cross section (having area of S), i.e. in a section perpendicular to the centreline, thus stresses in a single section containing the chosen point of the centreline. The **stress state**, however, was defined as a set of stresses in all sections which contain the chosen point. To judge the risk of failure (limit states), we need to know the stress state, i.e. stresses in all the sections containing the given point. stress state

To evaluate the stress state, we isolate an element as a free body, namely cut out by one cross section and one general (inclined) section, the normal line of which contains the angle with the centreline of the bar:

the area of this section will be $S_\rho = \dfrac{S}{\cos\alpha}$. The stress $\sigma = p$ acts in the cross section of the isolated element, and, in the general section ρ, a set of inner elementary area forces $f_\rho \mathrm{d}S_\rho$ acts; these elementary forces are also parallel to the x axis to satisfy the static equilibrium of the element.



$\vec{f}_\rho$ is the general stress and, because of the uniform stress state in the bar, we can assume that it is distributed uniformly throughout the section ρ. It results from the condition of static equilibrium:

uniform stress state

$$\sum F_x = 0: \qquad -\sigma S + f_\rho S_\rho = 0 \quad \Rightarrow \quad f_\rho = \frac{S}{S_\rho}\sigma = \sigma\cos\alpha.$$

We obtain the stress components by decomposing the general stress f_ρ into normal and tangential directions of the local coordinate system:

– normal stress: $\sigma_\rho = f_\rho \cos\alpha = \sigma\cos^2\alpha = \frac{\sigma}{2}(1+\cos 2\alpha)$,

– shear stress: $\tau_\rho = f_\rho \sin\alpha = \sigma \sin\alpha\cos\alpha = \frac{\sigma}{2}\sin 2\alpha$.

These relations express the dependency of the general stress f_ρ and its components σ_ρ and τ_ρ on the stress σ in the cross section and on the position of the section ρ towards the bar centreline. The stress state is therefore defined by the σ stress in the cross section, because it is possible to calculate stress in any section ρ from it.

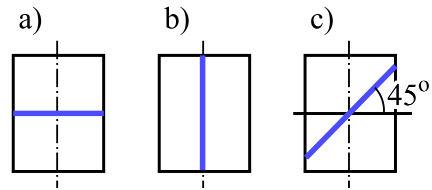
An analysis of these relations gives:

a) $\alpha = 0^o \quad \sigma_\rho = \sigma \quad \tau_\rho = 0$

b) $\alpha = 90^o \quad \sigma_\rho = 0 \quad \tau_\rho = 0$

c) $\alpha = 45^o \quad \sigma_\rho = \frac{\sigma}{2} \quad \tau_\rho = \frac{\sigma}{2} = \tau_{ex}$

here the extreme shear stress acts.



You can see that there are two sections in which the shear stress equals zero, namely the sections containing the angle of 0^o and 90^o with the centreline. Turning the section around any other axes in a 3D space, we could find some other sections having zero shear stress; one of them is directly the drawing plane in the figures. Evidently there exist three mutually perpendicular planes having zero shear stresses which are called **principal planes of the stress tensor**. The normal stresses in these planes are called principal stresses and denoted as $\sigma_1, \sigma_2, \sigma_3$; we order them with respect to their magnitude so that it holds $\sigma_1 \geq \sigma_2 \geq \sigma_3$. The directions of these principal stresses are given by the lines of intersection of the principal planes and they create the so called principal coordinate system. This system simplifies advantageously the stress tensor in the form

stress state

stress tensor

$$T_\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix},$$

in which the stress state is defined by the three independent principal stress components only. The remaining three components of the stress tensor define the orientation of the principal coordinate system what, however, is not substantial for judging the limit states at a homogeneous isotropic material. For the uniaxial tensile stress state in question it holds

$\sigma_1 = \sigma = \frac{N}{S}, \quad \sigma_2 = \sigma_3 = 0$, for the compressive one $\quad \sigma_1 = \sigma_2 = 0, \quad \sigma_3 = \sigma = \frac{N}{S} < 0.$

11.9.1. Graphical representation of stress state

This representation enables us to imagine the stress state and to calculate the extreme values of components of the general stress easily. For derivation of the graphical representation, we use the relations for the stresses in the general section ρ the normal line of which contains the angle α with the bar centreline.

$$\sigma_\rho = \sigma \cos^2 \alpha, \quad \tau_\rho = \frac{\sigma}{2} \sin 2\alpha.$$

We manipulate the equations to get

$$\sigma_\rho = \sigma \frac{1 + \cos 2\alpha}{2} = \frac{\sigma}{2} + \frac{\sigma}{2} \cos 2\alpha \quad \Rightarrow \quad \sigma_\rho - \frac{\sigma}{2} = \frac{\sigma}{2} \cos 2\alpha, \qquad \tau_\rho = \frac{\sigma}{2} \sin 2\alpha$$

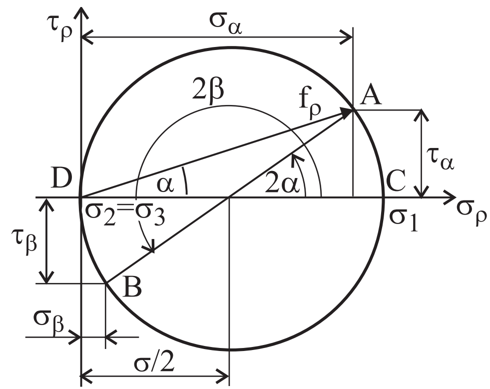
Then we bring the equations to a square and summarise both of them:

$$\begin{aligned}
\left(\sigma_\rho - \frac{\sigma}{2}\right)^2 + \tau_\rho^2 &= \left(\frac{\sigma}{2} \cos 2\alpha\right)^2 + \left(\frac{\sigma}{2} \sin 2\alpha\right)^2 \\
\left(\sigma_\rho - \frac{\sigma}{2}\right)^2 + \tau_\rho^2 &= \left(\frac{\sigma}{2}\right)^2 (\cos^2 2\alpha + \sin^2 2\alpha) \\
\left(\sigma_\rho - \frac{\sigma}{2}\right)^2 + \tau_\rho^2 &= \left(\frac{\sigma}{2}\right)^2
\end{aligned}$$

There are only two variables in the equations (σ_ρ *and* τ_ρ) which can be used as a basis of a coordinate system; this coordinate system defines the Mohr's plane of stresses.

Mohr's plane of stresses is a plane the abscissa and the ordinate of which have a meaning of normal σ_ρ and shear τ_ρ stresses, respectively, acting in a certain section ρ containing the point in question.

The derived equation represents a circle $((x - m)^2 + (y - n)^2 = r^2)$ in the Mohr's plane of stresses, having its center on the abscissa (σ_ρ axis) in the distance of $\frac{\sigma}{2}$ from the origin and radius $r = |\frac{\sigma}{2}|$. This circle is called **Mohr's circle of stresses for the simple tension** ($\sigma > 0$) or **compression** ($\sigma < 0$). A point of the Mohr's circle (having coordinates σ_ρ and τ_ρ) represents the general stress f_ρ in the point of the planar section defined by the angle α. Thus the whole circle **represents stresses in all sections** containing the given point,



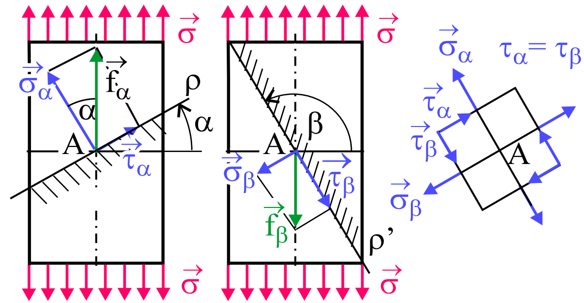
i.e. it represents the **state of stress** in this point under conditions of simple tension (compression). It is important to remember that the radius vector of a point of the Mohr's circle (with its beginning in the center of the circle) circumscribes the angle 2α which is double of the angle α contained by the normal line of the section and the bar centreline (this fact results from the derivation where the angle 2α occurs).

The following statements result evidently from the Mohr's circle:

- The points of intersection of the Mohr's circle with the abscissa (horizontal axis) determine the magnitudes of principal stresses. In the point C ($2\alpha = 0^o$, the plane section perpendicular to the centreline) and in the point D ($2\alpha = 180^o$, i.e. the plane section parallel to the centreline) the shear stresses equal zero.
- The maximum shear stress $\tau_{\max}$ is in the section containing the angle of 45^o ($2\alpha = 90^o$) with the centreline and equals $\tau_{\max} = \frac{\sigma}{2}$.

– Shear stresses in two diametric points A, B of the Mohr's circle are equal in magnitude but with opposite signs. In the real bar, components of stresses in two mutually perpendicular plane sections ρ (determined by the angle α) and ρ' (determined by the angle β – see the figure) correspond to these points.

We can obtain the same conclusion also in an analytical way. We introduce the section ρ cutting the centreline in the chosen point and defined by the angle α and, perpendicular to it, another section ρ' defined by the angle β.



$$\beta = \frac{\pi}{2} + \alpha, \quad 2\beta = \pi + 2\alpha,$$

$$\sin 2\beta = -\sin 2\alpha,$$

$$\tau_\beta = \frac{\sigma}{2}\sin 2\beta = -\frac{\sigma}{2}\sin 2\alpha = -\tau_\alpha$$

This relation formulates the **theorem of equality of shear stresses**:

equality of shear stresses

Shear stresses in two mutually perpendicular sections containing a certain point of the body are equal in magnitude and have directions such that either both of them point towards or both point away from the line of intersection of the sections.

The conclusions and equations derived from the Mohr's diagram of stresses are in contradiction to the conclusions based on the equations of static equilibrium of the element in point of the signs. This contradiction is attributed to the Mohr's representation itself, because the shear stresses in Mohr's diagram (given by ordinates of two diametric points

convention

in the Mohr's circle, thus equal in magnitude) have opposite signs. This contradiction requires to introduce a special sign convention concerning signs of the shear stresses in Mohr's representation:

The shear stress is taken as positive if it has the orientation of the outer normal line of the section $\vec{e}_n$ rotated by 90^o clockwise.

11.10. Fields of applicability of the simple tension of bars

The theory of simple elasticity of bars is based on

- bar assumptions,
- equations of static equilibrium of an element in its undeformed state.

These two basic assumptions made it possible to derive the above simple formulas describing stresses and deformations of bars. When using the theory of simple elasticity of bars to solve practical problems, it is important to judge its applicability. This examination requires broad knowledge because the basic assumptions are disturbed to a certain extent in nearly all cases. Therefore we limit ourselves to a qualitative examination of its applicability in this course, especially from the viewpoint of satisfaction of bar assumptions.

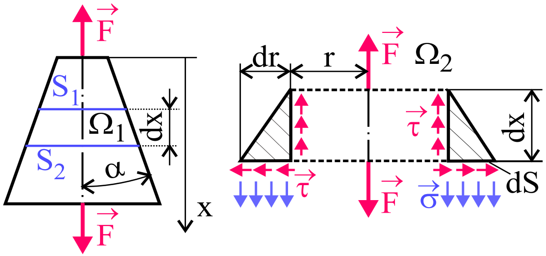
bar assumptions

11.10.1. Influence of the cross section variability along the centreline

a) Continuously varying cross section

Let's have a straight bar with a circular cross section changing continuously along the bar centreline. The conicity of the bar is determined by the angle α contained by the surface straight line and the axis of the cone. The bar is loaded by two equal forces $\vec{F}$ on both of its ends, that means the constant normal force of magnitude $N = F$ is the only one non-zero component of inner resultants along all the length of the centreline and the bar is loaded in tension. We isolate a onefold infinitesimal element Ω_1 as a free body from the bar by introducing two adjacent cross sections in the mutual distance of $\mathrm{d}x$.

Subsequently we isolate an element Ω_2 from it by a cylindrical section having its base area S_1 and being coaxial with the bar. There are normal stresses σ acting on the face of the element Ω_2. To achieve the static equilibrium of this element, shear stresses τ must act on the cylindrical section.



Because of the theorem of shear stresses, shear stresses of the same magnitude will act also in the cross sections. If we suppose, similarly to the prismatic bar, a uniform distribution of normal stresses σ throughout the cross section S_2 or $\mathrm{d}S$, we can obtain the equation of static equilibrium in the form

$$\sum F_x = 0 : \qquad \sigma \mathrm{d}S - \tau 2\pi r \mathrm{d}x = 0$$

The area $\mathrm{d}S$ can be expressed approximately as $\mathrm{d}S = 2\pi r \mathrm{d}r$, so that we obtain

$$\sigma 2\pi r \mathrm{d}r = \tau 2\pi r \mathrm{d}x \qquad \Rightarrow \qquad \frac{\tau}{\sigma} = \frac{\mathrm{d}r}{\mathrm{d}x} = tg\,\alpha$$

Thus the shear stress is proportional to the relative change of the bar thickness; at a conical bar, this relative change is expressed by the ratio $\mathrm{d}r/\mathrm{d}x$.

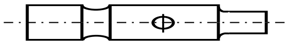
Because of the influence of the shear stresses acting in the cross sections, the uniaxiality of the tensile stress state is violated with the consequence of **distortion** (warping) of the cross sections. Thus the cross sections do not remain planar and the bar assumptions are not satisfied exactly. The larger are the changes in bar cross sections, the larger ar the deviations from the bar assumptions. If it holds $\mathrm{d}r/\mathrm{d}x < 0, 1$, i.e. the **angle of conicalness α** of the bar is less than 0,1 rad ≈ 6°, the shear stress is more than one order smaller in magnitude than the normal stress and the deviation from the bar assumptions can be neglected. The order of the limit value of conicity holds also for other shapes of cross sections.

bar assumptions

Warning! We derived the shear stresses under assumptions that the cross sections remain planar during the loading process. They, however, do not remain planar as a consequence of the acting shear stresses so that even the derived relations for evaluation of the shear stresses do not hold exactly.

b) Stepwise changes in cross sections (notches)

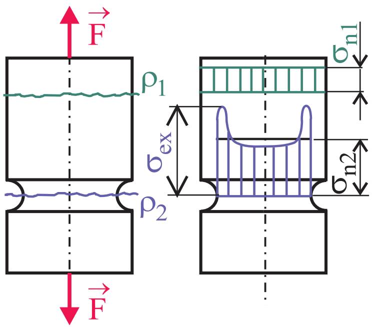
notches:
- structural - created **intentionally** by designer (fillets, necks, holes, shoulders)
- imperfections - consequence of the real production (inclusions, bubbles, cracks)



Most of the fractures during operation occur in locations of notches. It was found out that

- notches cause a local strain concentration and, consequently, a stress concentration (in the near surroundings of the notch the stress distribution is different from that supposed by the theory of simple elasticity);
- the **state of stress** changes in this location, **and** a **general triaxial stress state occurs** in the surroundings of the notch;
- the smaller is the notch radius, the higher the extreme stress in its root;
- the effect of the notch has a strictly local character.

A methodology was created which enables the designer to evaluate the stresses in the notches using the theory of simple elasticity of bars; the extreme stress value σ_{ex} in the root of the notch is calculated from the nominal stress σ_n using a **stress concentration factor** $\alpha = \sigma_{ex}/\sigma_n$.

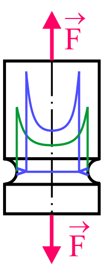


The nominal stress $\sigma_n = N/S$ is calculated using the relations valid for simple tension (compression), i.e. from the assumption of a uniform stress distribution throughout the cross section even in the notch location. stress

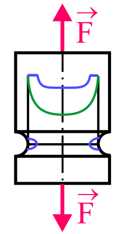
The values of stress concentration factors have been evaluated using computational (FEM) or experimental (photoelasticimetry) modelling, namely for various shapes of bars and notches and various types of loading, and the results have been represented in graphs.

It is usual that every graph contains the relation for calculation of the nominal stress σ_n in the section that the stress concentration factor α is related to. α graphs

In the analysis of limit states it is necessary to distinguish whether the material is in ductile or in brittle state. σ_{Rt}



If the material of a notched bar is in **brittle** state, then if it holds $\sigma_{ex} = \sigma_{Rt}$, a brittle crack occurs in the location of the stress concentration. It increases the stress concentration because its radius of rounding is lower than that of the original structural notch. Then it comes to an uncontrolled crack propagation under load until the **brittle fracture** occurs. Therefore the state when the stress peak reaches the ultimate stress of the material must be avoided.



σ_K

If the material of the bar is in **ductile** state, then **plastic deformations** occur in the location of stress concentration after achieving the equality $\sigma_{ex} = \sigma_K$. The plastic deformations reduce the stress concentration (the stress cannot extend significantly beyond the yield limit) and increase the strain concentration - the tip of the notch becomes blunter.

The influence of the notch should be always judged from the viewpoint of the possibility of brittle behaviour. The existence of the triaxial stress state in the notch surroundings can induce a brittle fracture even at the material being ductile under standard tension test conditions (smooth specimen).

safety factor

If the brittle fracture can be negated thanks to analysis and experience, low safety factors can be chosen, because there is another reserve of bearing capacity of the bar in the plastic region. However, the loading of most machine components is not static but variable in time; this repeated plastic deformation can cause a fatigue fracture. Evaluation of the fatigue fracture risk, however, requires application of other computational procedures that are not included in this course. In the cases of a significant brittle fracture risk, higher values of the safety factor against the brittle fracture use to be chosen.

Notch (as a stepwise change of the cross section)

– is **substantial** from the viewpoint of stress state and failure,
– is **mostly not substantial** from the viewpoint of deformation parameters of the bar.

deformation parameters

11.10.2. Influence of a screw-shaped bar

The bar is said to be screw-shaped (helical) if the central principal axes of the cross sections are not mutually parallel. A screw-shaped bar can be created by translation of a defined cross section along the bar centreline under its synchronous rotation. The helicality can be quantified by means of the ratio $\mathrm{d}\varphi/\mathrm{d}x$, where φ is the angle of rotation of principal axes of the cross section in relation to the initial section. **Analogically to conicity, helicality creates shear stresses** in the cross sections. If the change in position of the neighbouring cross sections (quantified by means of the ratio $\mathrm{d}\varphi/\mathrm{d}x$) is sufficiently low, then the shear stress is negligible in comparison to the normal stress ($\tau \ll \sigma$) and formulas valid for simple tension can be used.

principal axes

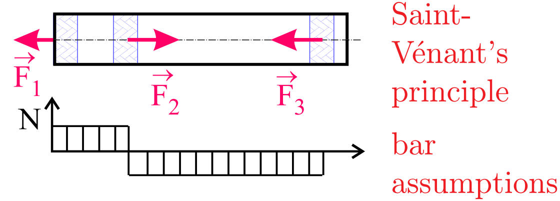
11.10.3. Variability of normal force along the centreline of a straight bar

The variability of normal force can be caused by either isolated forces or distributed loads (volume forces).

a) Loads in isolated sections

If there are isolated loads acting in the bar axis, the applicability of the model is restricted as follows:

- the bar assumptions are not satisfied but in a sufficient distance (in the sense of Saint-Vénant's principle) from the points of action of the isolated loads; in their surroundings the stress state is always non-uniform,
- in practice, the action of isolated loads is possible only either through existence of a notch (hole, groove, shoulder) or under violation of assumptions on bar-type stress state (gripping in pincers - the pressure in transversal direction cannot be comprehend into the theory of bars).

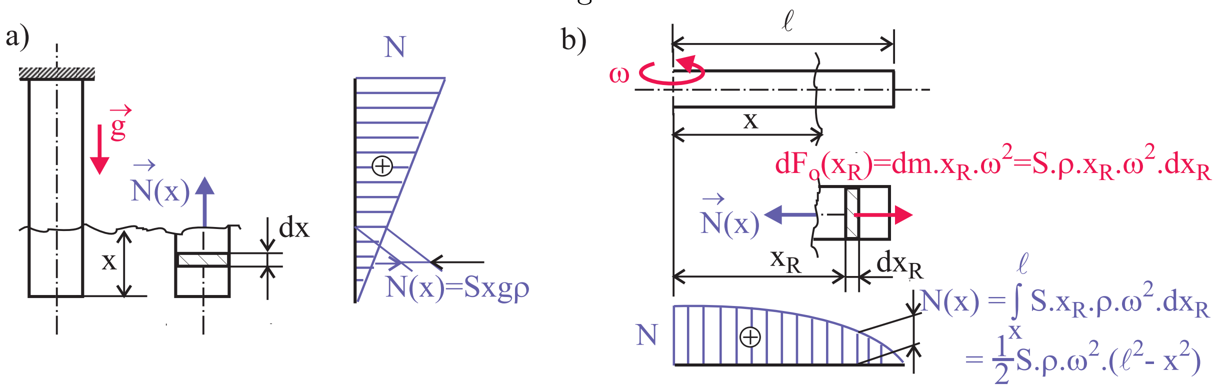


b) Distributed load by volume forces

Let's have a prismatic bar in a force field with its intensity parallel to the bar centreline.

Practical applications: a) vertical bar ⇒ gravitational field,
b) a bar rotating around an axis perpendicular to its centreline ⇒ field of centrifugal forces.

Problem 402
Example 404



The normal force and stress vary along the centreline but the stress distribution is uniform throughout the cross sections. Therefore the simple theory of elasticity of bars is applicable (shear stresses are not substantial).

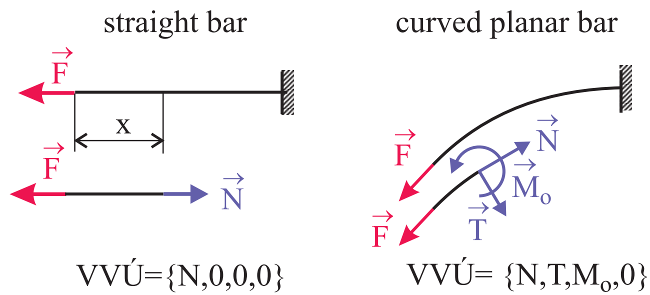
Stresses, displacement of point R of the centreline and strain energy of the bar of the length l must be calculated using the relations valid for bar with varying normal force:

$$\sigma(x_R) = \frac{N(x_R)}{S}, \quad u(x_R) = \int_0^{x_R} \frac{N(x)}{ES}\mathrm{d}x, \quad W(l) = \int_0^{l} \frac{N^2(x)}{2ES}\mathrm{d}x.$$

11.10.4. Curvature of the bar centreline

Let's have a bar the centreline of which is a continuous and smooth curve. The character of loading of the bars with curved centreline depends on

- the shape of bar centreline (type of the curve, planar or 3D, opened or closed),
- the relation between the radius of curvature and the characteristic dimension of the cross section (bars of low or high curvature),
- the type of the force system acting on the bar.



curved bar

combine loading

The type of loading is always combined at a curved bar, simple tension cannot occur.

However, there is a curved bar that can be solved approximately using the theory of simple tension – a planar thin-walled ring (annulus) under axisymmetric loads. There are two types of possible loads:

a) a uniform pressure on the inner or outer surfaces (e.g. a ring forced on a shaft with an interference),
b) centrifugal forces - a rotating ring.

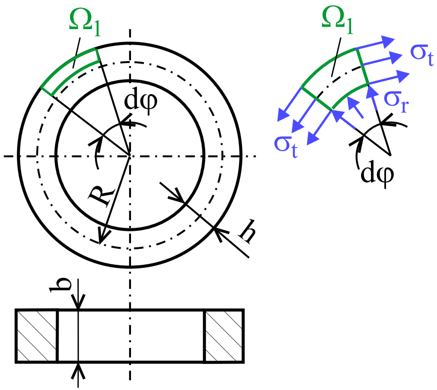
Problem 405

Problem 413

Problem 412

In both cases, the problem is axisymmetric. There are axisymmetric stress components acting on the basic element Ω_1 isolated as a free body from the ring:

- circumferential stresses σ_t in the cross sections,
- radial stresses σ_r in the cylindrical sections.



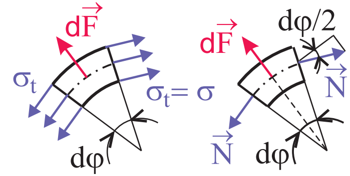
In thin-walled rings ($h \ll R$), the radial stress σ_r can be neglected in comparison with the circumferential stress σ_t. Then the only one significant stress component is σ_t, and it is distributed approximately uniformly throughout the cross section; there is an approximately uniform uniaxial stress state in the bar, similar to the simple tension.

uniform
stress state

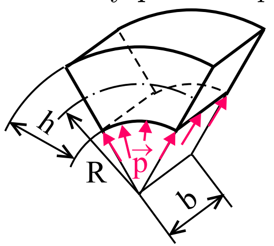
tension

The procedure of solution to both cases is similar, only centrifugal forces (if they are significant) must be included (in the sense of d'Alembert's principle) in the equilibrium equations of the element Ω_1 in radial direction:

$$\mathrm{d}F - 2N\sin\frac{\mathrm{d}\varphi}{2} = 0 \qquad \sin\frac{\mathrm{d}\varphi}{2} \doteq \frac{\mathrm{d}\varphi}{2} \Rightarrow N = \frac{\mathrm{d}F}{\mathrm{d}\varphi}$$

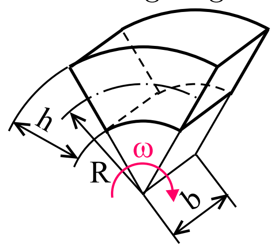


loaded by pressure $\vec{p}$



$$\mathrm{d}F_p = p\mathrm{d}S = pbR\mathrm{d}\varphi$$
$$N_p = \frac{\mathrm{d}F_p}{\mathrm{d}\varphi} = pbR$$
$$\sigma_{tp} = \frac{N_p}{S} = \frac{N_p}{bh} = \frac{pR}{h}$$

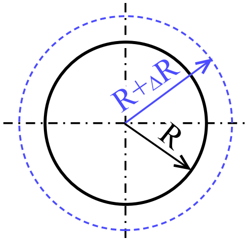
rotating ring



$$\mathrm{d}F_o = \mathrm{d}ma_\omega = \rho R\mathrm{d}\varphi bh\omega^2 R$$
$$N_o = \frac{\mathrm{d}F_o}{\mathrm{d}\varphi} = \rho R^2 bh\omega^2$$
$$\sigma_{to} = \frac{N_o}{S} = \frac{N_o}{bh} = \rho(R\omega)^2$$

The change in radius R of the centreline (displacement in **radial** direction) can be calculated from the **circumferential** strain (it is uniform around the ring because of the axisymmetry):

$$\varepsilon_t = \frac{2\pi(R+\Delta R) - 2\pi R}{2\pi R} = \frac{\Delta R}{R}$$



As the stress state is uniaxial, the simplified form of Hooke's law can be used:

$$\varepsilon_t = \frac{\sigma_t}{E} \quad \Rightarrow \quad \Delta R = R\frac{\sigma_t}{E}$$

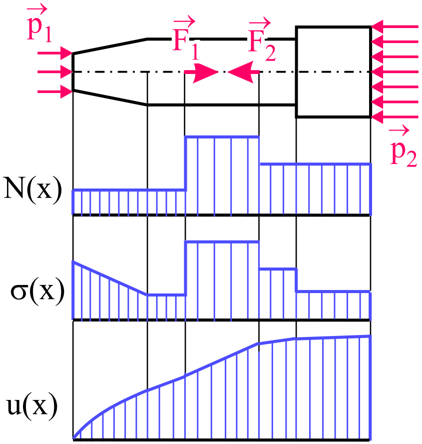
11.11. Solving problems concerning simple tension (compression) of bars

11.11.1. Free bar

We derived the relations for calculation of stresses, deformation parameters and strain energy, valid for a bar under simple tension (compression) if the bar assumptions are satisfied: bar assumptions

$$\sigma(x_R) = \frac{N(x_R)}{S(x_R)}, \quad u(x_R) = \int\limits_0^{x_R} \frac{N(x)}{ES(x)} \mathrm{d}x, \quad W(l) = \int\limits_0^{l} \frac{N^2(x)}{2ES(x)} \mathrm{d}x.$$

If the quantities $N(x), S(x)$ (and E, naturally) are constant along the bar centreline, the integration is trivial:



If $N(x)$ and $S(x)$ vary but not as much to violate substantially the assumptions of simple loading, the centreline should be divided into intervals in which each of these quantities (N, S, E) can be expressed by a single relation. The boundaries of the intervals are then in the points where a change occurs in any of the functions describing inner resultants, material properties or cross sections. The displacement of a certain point is a sum of elongations of the individual intervals in which the centreline was divided between the point in question and the referential point.

These elongation can be caused by acting normal forces

$$u(x_R) = \int_0^{x_R} \frac{N(x)}{ES(x)} \mathrm{d}x.$$

If there is a temperature load, the thermal dilatation must be added

$$u_T(x_R) = \alpha \Delta T x_R,$$

where α is the coefficient of thermal expansion.

In this course, we restrict ourselves to the following ones of all the possible **limit states**:

- **limit state of deformation** - the deformation of the body (structure) becomes inadmissible from the viewpoint of its function, the limit values are given by limit displacements or limit rotations of some points of the centreline. The safety factor against this limit state is defined by the ratio of the limit to the operation values of the deformation parameter in question $k_D = \frac{u_M}{u}$ or $k_D = \frac{\varphi_M}{\varphi}$. limit state
- **limit state of elasticity** - in this state first macroplastic deformations occur, yield stress σ_K is the corresponding limit value and is evaluated from the results of tension test. The safety factor against the limit state of elasticity is $k_K = \frac{\sigma_K}{|\sigma|}$. limit state of elasticity

safety factor

The safety factor k_K against the limit state of elasticity is related to one point of the bar only. In general, its value is different in each of the points of the bar, therefore

$$k_K = k_K(x, y, z), \quad (x, y, z) \in \Omega.$$

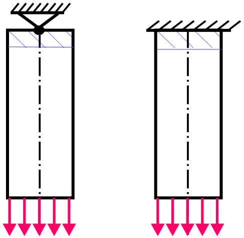
To judge the reliability of the bar, it is necessary to find the point where the safety factor k_K is minimal, and then the cross section containing this point. The following terms are used for them:

dangerous point of the bar - the point with minimal value of the safety k_K,
dangerous section of the bar - the cross section containing the dangerous point.

The safety factor of the bar equals then the safety factor evaluated in its dangerous point (the minimal value of the safety factors in all the points of the bar).
As the σ stress distribution across the section is uniform in the case of simple tension (compression), all points of the dangerous cross section are dangerous points.

11.11.2. Supported bar

There is a region in the near surroundings of the support where the assumptions of the simple tension are not satisfied (in practice, for instance, it is not possible to support the bar in the points of its centreline only). The extreme stresses in this region are higher than those calculated using the theory of simple tension. If we need a more precise evaluation of these stresses, we can use e.g. finite element method.



bar assumptions

FEM

Procedure of solving supported bars

1. We isolate the bar as a free body and introduce reactions in the locations of the removed supports.
2. We formulate the applicable equations of static equilibrium. The only non-trivial condition of static equilibrium is the force condition in the direction of x axis ($\sum F_x = 0$).
3. We evaluate the degree of statical indeterminacy (redundancy) $s = \mu - \nu$. The following cases can occur: redundancy
   a) $s = 0$ – the bearing of the beam is **statically determinate** – we continue with par. 7, par. 4 – 6 can be omitted. Problem 403
   b) $s \geq 1$ – the bearing of the beam is **statically indeterminate** – we continue with par. 4.
4. We create a released structure and formulate the compatibility equations; in the case of simple tension (compression) of bars, they are given by the displacement values of such many centreline points, the number of which must equal the degree of redundancy. released structure

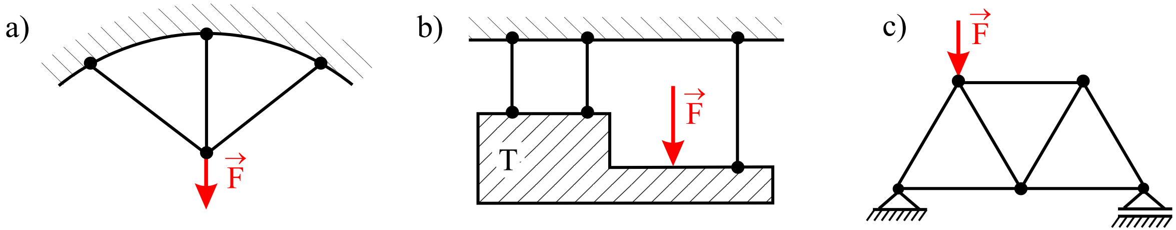
5. We express the compatibility equations by means of loads and temperature changes, with taking the production inaccuracies into account (clearances or interferences). The compatibility equations can be:
   a) homogeneous – the restricted kinematic parameter equals zero, Example 414
   b) non-homogeneous – the restricted kinematic parameter differs from zero because of the production inaccuracies (e.g. an assembly clearance or interference cleared down before welding) or thermal dilatations, Example 417 Example 418
   c) circumstantial – the bar can be either statically determinate or indeterminate in dependence on the magnitude of displacement (e.g. an assembly clearance disables the function of the support). We should determine which of these situations comes into being. Example 437 Problem 408

   Note: if the body (structure) is supported by rigid as well as flexible supports, the released structure should preserve the rigid support and the flexible ones should be removed and replaced by reactions, and non-homogeneous compatibility equations should be formulated for them; otherwise the position of the body is not fixed in the released structure and a problem occurs of how to distinguish the deformation displacements from the movement of the whole body. deformation
6. We formulate a set of equations consisting of conditions of static equilibrium of the beam isolated as a free body and compatibility equations of the released structure. It is needed to express the deformations by means of loads using the formulas for displacements of centreline points or Castigliano's theorem. displacement Castigliano's theorem
7. We solve the set of equations for support reactions.
8. We solve the stress state and deformation parameters in the same way as for a free beam.

11.11.3. Systems with bars loaded by simple tension (compression)

Models (abstract systems) with bars can be divided into three groups:

a) systems consisting only of bars, each of them being coupled with the base by pin supports, supports
b) systems consisting of bars coupled by pin supports to a rigid body (deformations of which are negligible against the deformations of the bars),
c) strut frames, as computational models of lattice-works (railway bridges, cranes etc.). strut frame



In practice, all of these systems are made with fixed supports, not with the pin supports. The computational model based on pin supports can be used for them if and only if the moments in supports are negligible. This is the case under the following conditions:

– the bars (struts) are long and slender (i.e. the length at least one order higher than the thickness),
– loads act only in framework joints or on a rigid body (to avoid a significant bending),
– the system remains immovable after introduction of pin joints (i.e. a statically determinate or indeterminate system).

Under the above conditions each of the bars is a unloaded binary member (i.e. having just two supports); all the supports of the strut frame are modelled as pin supports (or as spherical supports in a 3D structure). The static equilibrium of any of such bars results in the fact that both of the reactions must equal in magnitude and their lines of action must be identical with the bar centreline. Thus they represent an only one unknown parameter, equal to the normal force in the bar. Under these conditions the bars are loaded only in tension (if $N > 0$) or in compression (if $N < 0$).

Because of the above simplifications, free body diagrams of bars are not more needed for the solution and we isolate only framework joints as free bodies.

a) **Systems of bars coupled with the base**
We isolate only the joint as a free body. As we suppose tension in each of the bars (the forces act outwards every bar), the positive forces will act outwards the joint (according the principle of action and reaction).

i) **Statically determinate system**
The set of equilibrium equations can be solved for the unknown forces which equal the normal forces in the individual bars.

Example 422

Problem 415

ii) **Statically indeterminate system**
We proceed according to the general procedure for solutions to statically indeterminate problems, as it is presented for a single bar in the previous chapter.

Example 426

Example 427

Example 430

Problem 409

If we know the normal forces in the bars, we can calculate stresses, strain energy of the system or displacements of the individual framework joints. To calculate the displacements, it is necessary to use Castigliano's theorem in nearly all cases; the strain energy must be expressed for the whole strut frame.

stress

strain energy

displacement

Castigliano's theorem

The strain energy of an i th bar of the length l_i loaded by simple tension is

$$W^{(i)} = \int_0^{l_i} \frac{N_i^2(x)}{2E_i S_i(x)} \mathrm{d}x.$$

Thus, the displacement u_J of the point of action of the force $\vec{F}_J$ (which acts on a single bar of the length l_i) in the direction of this force equals

$$u_J = \frac{\partial W}{\partial F_J} = \int_0^{l_i} \frac{N(x)}{ES(x)} \frac{\partial N(x)}{\partial F_J} \mathrm{d}x.$$

Since the characteristics $N(x), S(x)$ and $E(x)$ are constant along each of the bars of the strut frame in question, the strain energy of the system consisting of n bars equals

$$W = \sum_{i=1}^{n} W^{(i)} = \sum_{i=1}^{n} \frac{N_i^2 l_i}{2E_i S_i}$$

and the displacement u_J of the point of action of the isolated force $\vec{F}_J$ in its direction is

$$u_J = \frac{\partial W}{\partial F_J} = \sum_{i=1}^{n} \frac{N_i l_i}{E_i S_i} \frac{\partial N_i}{\partial F_J}.$$

b) **Systems consisting of bars and rigid bodies**

i) **Statically determinate system**

The set of equilibrium equations of rigid bodies (the deformation of which is negligible) can be solved for the unknown reactions which equal to the normal forces in the individual bars. Example 434

ii) **Statically indeterminate system**

We create free body diagrams of the rigid bodies only. The released structure for formulation of compatibility equations can be arbitrary but the framework joints being supported by the base can be removed advantageously; deformations of the base are negligible so that the corresponding kinematic support parameters equal zero. The compatibility equations can be homogeneous, non-homogeneous or conditioned. Example 435 Problem 410 compatibility equations

If we know the normal forces in the individual bars, we can calculate stresses in the bars, strain energy of the system or displacements of any points of the system. stress strain energy displacement

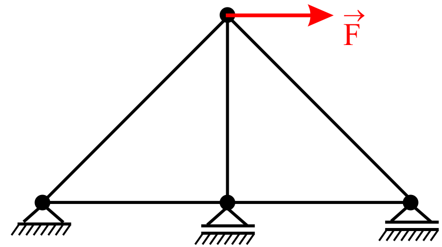
c) **) Strut frames** strut frame

i) **Statically determinate system**

If the strut frame is internally as well as externally statically determinate, we solve the system of equations (of static equilibrium of framework joints) for the unknown forces in the bars using the successive or general joint method. Example 308 Problem 416 Problem 420 Problem 421

ii) **Statically indeterminate system**

To solve the normal forces in the bars, we need additionally the compatibility equations, based on the released structure.



released structure

Problem 424

Problem 428

Problem 429

Problem 425

Problem 431

a) externally statically indeterminate

static analysis:

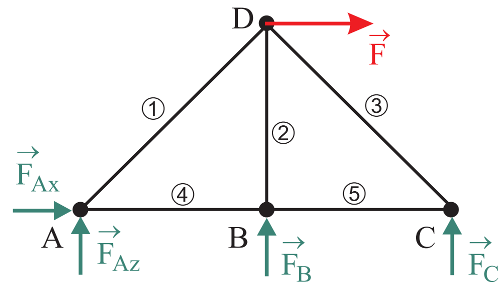
$\mu_{ex} = 4, \quad \nu = 3$

$s_{ex} = \mu_{ex} - \nu = 4 - 3 = 1$

$s_{in} = p - (2k - 3) = 5 - (2 \cdot 4 - 3) = 0$

The problem is externally onefold statically indeterminate and internally statically determinate.

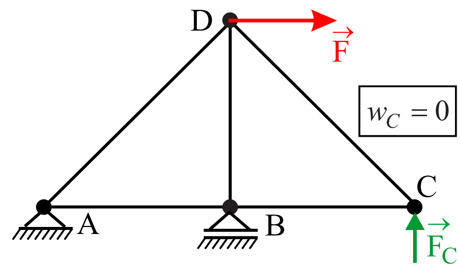
The free body diagram can be created by isolation of the strut frame from the base.



statický rozbor

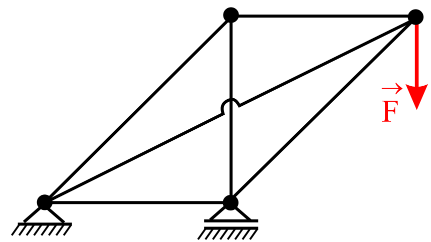
Example 302

We create a released structure (i.e. a statically determinate structure) of the strut frame (which can be understood as a body consisting of bars, the so called bar body, if the bars are mutually immovable) and formulate the compatibility equations (homogeneous, non-homogeneous or circumstantial).



released structure

b) internally statically indeterminate



static analysis:

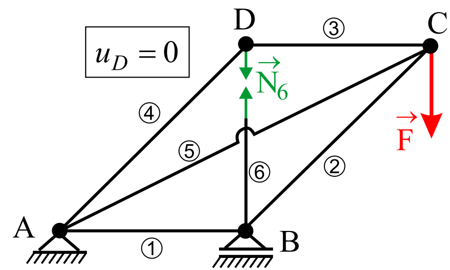
$\mu_{ex} = 3, \quad \nu = 3$

$s_{ex} = \mu_{ex} - \nu = 3 - 3 = 0$

$s_{in} = p - (2k - 3) = 6 - (2 \cdot 4 - 3) = 1$

The problem is externally statically determinate and internally onefold statically indeterminate.

The creation of a released structure consists of:



compatibility equations

Problem 423

– releasing of s_{in} bars in a joint,
– introducing of a normal force in the end of the released bar and of a contrary force of the same magnitude in the joint where the bar was fixed (principle of action and reaction),
– formulation of the compatibility equation(s) in the point(s) where the bar has been released; this equation expresses the mutual displacement of both of the released points.

c) internally and externally statically indeterminate It is a combination of both of the previous types of redundancies, compatibility equations of both types must be formulated.

Example 303

Example 436

Now the system of equations of static equilibrium of joints and of compatibility equations can be solved for normal forces in the bars independently of the type of redundancy and, consequently, stresses in the bars can be calculated. The displacements of joints can be calculated by means of Castigliano's theorem from the total strain energy of the whole structure.

stress

Castigliano's theorem

11.12. Examples and problems

Examples

Problem 404	Problem 414	Problem 417	Problem 418	Problem 419
Problem 422	Problem 426	Problem 427	Problem 430	Problem 432
Problem 433	Problem 434	Problem 435	Problem 436	Problem 437

Problems

Problem 401	Problem 402	Problem 403	Problem 405	Problem 406
Problem 407	Problem 408	Problem 409	Problem 410	Problem 411
Problem 412	Problem 413	Problem 415	Problem 416	Problem 420
Problem 421	Problem 423	Problem 424	Problem 425	Problem 428
Problem 429	Problem 431			