10. Simple elasticity theory of bars

The objective of mechanics of materials is the solution to problems related to stress states, deformations and failures of components of technical objects; these components are often very complex in shape. The evaluation of stresses and strains in bodies with a complex shape was made possible by use of modern computers and numerical methods. Earlier, only solutions to some bodies with a simpler geometry were possible, namely under many further limitations.

limitations

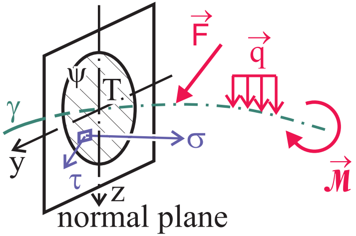
Now we start with the simplest model body – a **bar**. In a common language a bar is understood as a „long and slender“ body. Here it is necessary to define the bar more preciously; it is a generalisation of model bodies such as beam, column, shaft, strut etc., each of which represents a special case of a bar, as it will be specified later. The following definition introduces some additional limitations but, on the other hand, it enables us to comprehend also some bodies which are not „long and slender“ (if they satisfy these limitations).

> In the mechanics of materials, bar is the simplest theoretical model of a real body which satisfies certain assumptions concerning geometry, deformations, loads, supports and stress states. All these assumptions will be called **bar assumptions**.

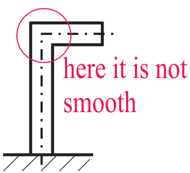
10.1. Bar assumptions

a) assumptions concerning geometry

– The bar is defined by its centreline γ and by the cross section ψ in every point of this centreline.

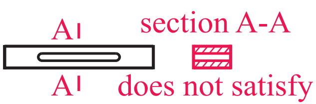


– The centreline γ is a continuous and smooth line with a finite length.



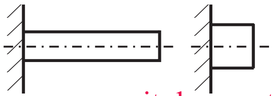
– The cross section is a onefold or multifold **continuous** plane region, defined by its outlines; it can be described mathematically by its cross section characteristics.

characteristics



The section A-A in the figure is an example of an **discontinuous** cross section (violation of bar assumptions, the body cannot be solved as a bar).

– The centreline length is substantially higher then the largest dimension of the cross section.

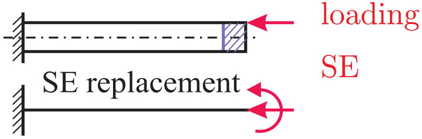


A cartesian right-handed coordinate system is mostly used for the description of a bar; its x axis is tangential to the bar centreline and the other two axes (y, z) lie in the cross section.

b) assumptions concerning supports and loads

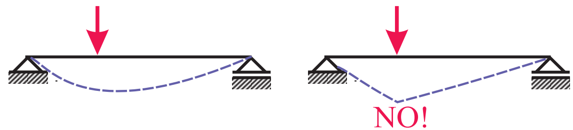
– The supports restrict only displacements and rotations of points belonging to the centreline.

– The loads are concentrated on the centreline, i.e. the bar can be loaded by isolated or line forces or couples which act on the centreline. If this is not satisfied, it is necessary to introduce a statically equivalent (SE) replacement of the real load by a load acting on the centreline; the limitations based on Saint-Venant's principle must be taken into account, if using a SE replacement.



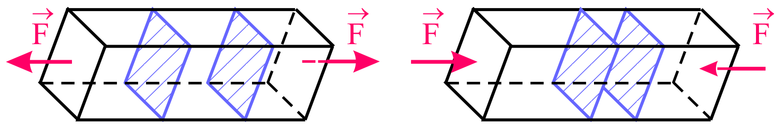
c) assumptions concerning deformation

– The centreline remains continuous and smooth during all the process of deformation.



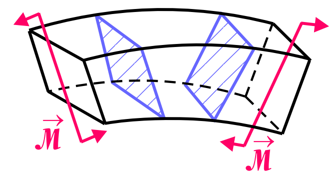
– During all the process of deformation, the **cross sections remain planar** and perpendicular to the deformed centreline, they can only mutually

  – draw away (tension), draw near (compression),



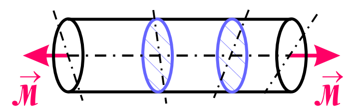
tension

  – rotate around an axis lying in the cross section and deform (flection),



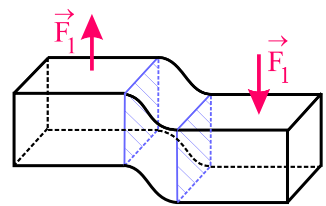
flection

  – rotate around an axis perpendicular to the cross section and remain undeformed (torsion),



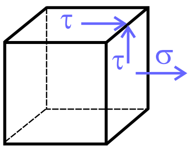
torsion

  – shift perpendicular to the centreline (shear).



d) assumptions concerning stress states

The state of stress in a point of the bar is determined by normal and shear stresses in the cross section containing this point; all the other stress components equal zero. This type of stress state is called **bar-type stress state.**

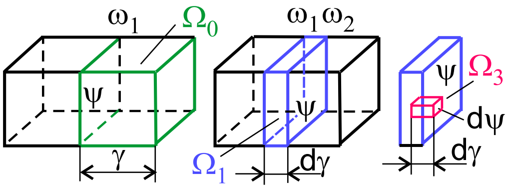


bar-type s.s.

$$T_\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad T_\sigma = \begin{pmatrix} \sigma_x & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{pmatrix}$$

Two types of elements will be used in solving stresses and deformations of bars:

- finite element Ω_0, isolated from the bar by a single section ω_1,
- onefold infinitesimal element Ω_1, isolated from the bar by two adjacent cross sections ω_1 and ω_2 (this is the **basic** element in bar solutions).

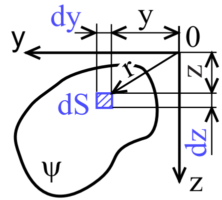


It is often advantageous to understand this basic element as a set of threefold infinitesimal elements, which are defined by an element dψ of the cross section ψ and by an element dγ of the centreline γ.

10.2. Geometrical characteristics of the cross section

Geometrical characteristics of the cross section are quantities that characterise the cross section and that are used in formulas for calculation of stresses and deformations under the specific simple types of loading.

10.2.1. Cross section area



$$S = \int_{\psi} \mathrm{d}S = \iint_{\psi} \mathrm{d}y\mathrm{d}z \qquad \left[\mathrm{m}^2\right]$$

10.2.2. Linear (static) moments

$$U_y = \int_{\psi} z\mathrm{d}S, \quad U_z = \int_{\psi} y\mathrm{d}S \qquad \left[\mathrm{m}^3\right]$$

You have met the linear moments already in statics where they were used in calculation of the position of gravity centre: gravity centre

$$\vec{r}_T = \frac{\int_{\Omega} \vec{r}\mathrm{d}F_G}{\int_{\Omega} \mathrm{d}F_G}$$

$$\text{for} \quad \rho = \text{konst.}, t = \text{konst.} \quad \vec{r}_T = \frac{\int_{\Psi} \vec{r}\mathrm{d}S}{S} \quad \Rightarrow \quad y_T = \frac{\int_{\Psi} y\mathrm{d}S}{S} = \frac{U_z}{S}, \quad z_T = \frac{\int_{\Psi} z\mathrm{d}S}{S} = \frac{U_y}{S}$$

Note: Example 101

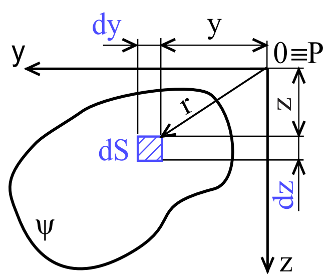
Linear moment to an axis containing the gravity centre (centroidal axis) equals zero.

10.2.3. Second moments (moments of inertia)

term	definition formula	dimension	example of use
axial	$J_y = \int\limits_{\psi} z^2 \mathrm{d}S,$ $J_z = \int\limits_{\psi} y^2 \mathrm{d}S,$	$[\mathrm{m}^4]$	stress and deformation in flection (calculated in the principal coordinate system)
deviation (product of inertia)	$J_{yz} = \int\limits_{\psi} yz \mathrm{d}S,$	$[\mathrm{m}^4]$	determination of the directions of principal axes
polar	$J_P = \int\limits_{\psi} r^2 \mathrm{d}S,$	$[\mathrm{m}^4]$	stress and deformation in torsion (for bars with axisymmetric cross sections)

flection

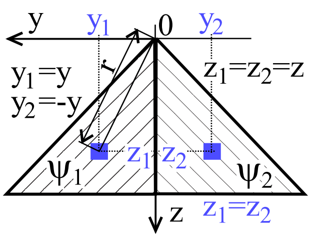
torsion



10.2.4. Basic properties of second moments (moments of inertia)

1. They are additive: second moments of the whole cross section ψ to the given axes equal the sum of second moments of the cross section parts ψ_i to the same axes.
2. Signs: Values of axial and polar second moments are positive; values of products of inertia can be any real number (both conclusions result from the properties of integrals).
3. Axial moments of two symmetric sections to the axis of symmetry are equal. The same holds for any axis perpendicular to the axis of symmetry of both sections. Products of inertia of both sections to these axes are also of the same magnitude but of opposite signs. Example 102

Demonstration:



$$\Psi_1 = \Psi_2,\ J_z^{(1)} = \int_{\Psi_1} y^2 \mathrm{d}S \ = \int_{\Psi_2} (-y)^2 \mathrm{d}S \ = J_z^{(2)}$$

$$J_y^{(1)} = \int_{\Psi_1} z^2 \mathrm{d}S \ = \int_{\Psi_2} z^2 \mathrm{d}S \ = J_y^{(2)},$$

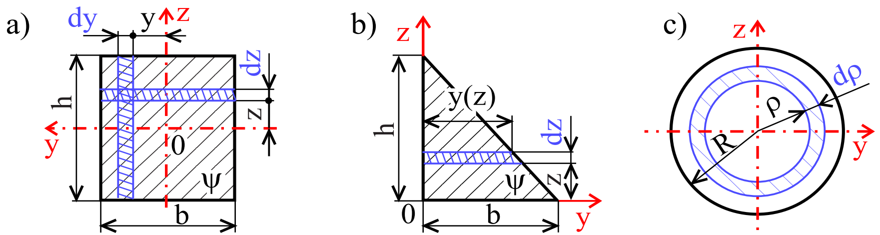
$$J_{yz}^{(1)} = \int_{\Psi_1} yz \mathrm{d}S \ = -\int_{\Psi_2} yz \mathrm{d}S \ = -J_{yz}^{(2)} \qquad \Rightarrow \qquad J_{yz}^{(1)+(2)} = J_{yz}^{(1)} + J_{yz}^{(2)} = 0.$$

It results from the above equations: if at least one of the coordinate axes is identical with the axis of symmetry of the cross section, the product of inertia of the cross section to this coordinate system equals zero.

4. The polar **second** moment equals the sum of axial **second** moments to the perpendicular axes intersecting each other in the pole.
   Demonstration:

$$r^2 = y^2 + z^2 \Rightarrow J_P = \int_{\psi} r^2 \mathrm{d}S = \int_{\psi} (y^2 + z^2)\mathrm{d}S = \int_{\psi} y^2 \mathrm{d}S + \int_{\psi} z^2 \mathrm{d}S = J_z + J_y$$

10.2.5. Second moments of basic simple cross section shapes



a) Rectangle $J_y = \int_\psi z^2 \mathrm{d}S = \int_{-h/2}^{h/2} z^2 b \mathrm{d}z = \frac{bh^3}{12}$, $J_z = \int_\psi y^2 \mathrm{d}S = \int_{-b/2}^{b/2} y^2 h \mathrm{d}y = \frac{hb^3}{12}$,

$J_{yz} = \int_\psi yz \mathrm{d}S = 0$

b) Triangle $\frac{y(z)}{b} = \frac{h-z}{h} \rightarrow y(z) = b - \frac{b}{h}z$, $\quad \mathrm{d}S = (b - \frac{b}{h}z)\mathrm{d}z$

$J_y = \int_\psi z^2 \mathrm{d}S = \int_0^h z^2 (b - \frac{b}{h}z)\mathrm{d}z = \frac{bh^3}{12}$, $J_z = \frac{hb^3}{12}$, $J_{yz} = \frac{h^2b^2}{24}$

Note: These moments must be transformed by *translation and rotation* for practical use, because they are not related to the principal **centroidal** coordinate system (see below).

c) Circle $J_y = J_z, \quad J_y + J_z = J_P \Rightarrow J_y = \frac{1}{2}J_P$, $\mathrm{d}S = 2\pi\rho \cdot \mathrm{d}\rho$, $J_P = \int_\psi \rho^2 \mathrm{d}S$,

$J_P = \int_0^R \rho^2 2\pi\rho \cdot \mathrm{d}\rho = \frac{\pi R^4}{2}$, $J_y = J_z = \frac{J_P}{2} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$, $J_{yz} = 0$

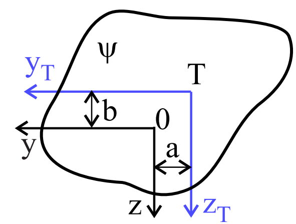
10.2.6. Second moments under transformation of coordinate system

The transformation relations can be advantageously used in calculation of **second** moments of the cross section. You can calculate the **second** moments related to the axes for which the calculation is easy (or the values are known), and then you transform the values to the *principal centroidal coordinate axes*. In this way you obtain the so called principal centroidal **second** moments that are used e.g. in calculations of stresses and deflections of bars under flection (beams). flection

a) Transformation by translation

Steiner's theorems Example 103 Problem 104

Using Steiner's theorems, we can calculate **second** moments to the translated axes y and z from the known values of **second** moments to the centroidal axes (intersecting each other in the gravity centre) y_T a z_T (or opposite):



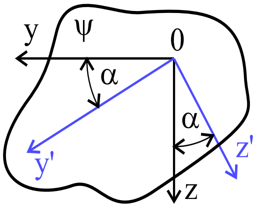
$$J_y = J_{y_T} + b^2 S, \quad J_z = J_{z_T} + a^2 S, \quad J_{yz} = J_{y_T z_T} + abS.$$

As the terms $a^2 S$ and $b^2 S$ are always positive, the **second** moment to any translated axis is higher than the **second** moment to the parallel centroidal axis (running through the gravity centre).

b) b) Transformation by rotation

Problem 110

For the transformation by rotation, the following relations can be derived:



$$\begin{aligned} J_{y'} &= J_y \cos^2\alpha - J_{yz}\sin 2\alpha + J_z \sin^2\alpha \\ J_{z'} &= J_z \cos^2\alpha + J_{yz}\sin 2\alpha + J_y \sin^2\alpha \\ J_{y'z'} &= \frac{J_y - J_z}{2}\sin 2\alpha + J_{yz}\cos 2\alpha \\ J_{P'} &= J_{y'} + J_{z'} = J_P \end{aligned}$$

10.2.7. Principal moments of inertia

Among the rotated coordinate systems y' and z', there is at least one coordinate system y_h, z_h in which the product of inertia equals zero ($J_{y_h z_h} = 0$). This coordinate sys-tem is the **principal coordinate system** and its axes are called **principal axes**. The second moments related to this coordinate system are called **principal** second **moments** (principal moments of inertia) J_{y_h} , J_{z_h} . It is evident from the Mohr's representation (see chapter 10.2.8.) that one of these second moments is maximal (denoted as J_1) and the other one minimal (J_2) among all the second moments to any rotated coordinate system. The position of the principal coordinate system is determined by the angle between the original and rotated (principal) axes; this angle can be calculated from the condition of zero product of inertia:

$$J_{y_h z_h} = \frac{J_y - J_z}{2} \sin 2\alpha_h + J_{yz} \cos 2\alpha_h = 0 \quad \Rightarrow \quad \alpha_h = \frac{1}{2} \text{arctg} \left(\frac{-2J_{yz}}{J_y - J_z} \right).$$

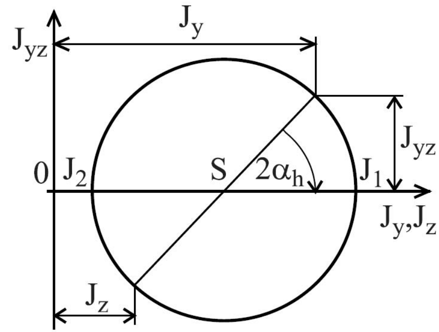
The principal coordinate system having its origin in the centroid of the cross section is called **principal** centroidal **coordinate system**. As the axis of symmetry of the section always runs through its gravity centre and the product of inertia to the centroidal axis equals zero, the axis of symmetry is always identical with the principal centroidal axis, as well as the perpendicular axis intersecting it in the centroid.

Example 105
Problem 107
Problem 108
Problem 109

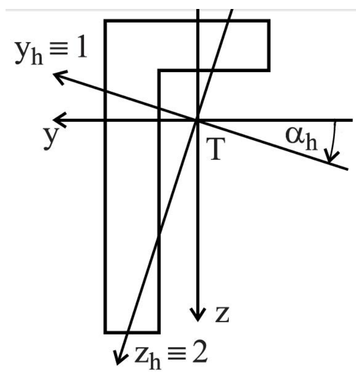
10.2.8. Mohr's representation of second moments

Second moments related to the coordinate systems having various rotation angles around a point of the cross section (commonly around the centroid) can be graphically represented in the form of the **Mohr's circle**.

In the Mohr's diagram, the axial second moments correspond to the abscissa and products of inertia to the ordinate of the graph. One point of the circle is determined by the axial second moment J_y and by the product of inertia J_{yz}, while the point on the opposite end of the circle diameter is determined by the axial second moment J_z and by the product of inertia J_{yz} of the same magnitude but with an opposite sign. (This convention is given by the derivation of the Mohr's circle.)



The angle $2\alpha_h$ between the radius vector of the point corresponding to the moment J_y and the abscissa (horizontal axis) equals the double angle α_h between the y-axis and the corresponding principal centroidal axis of the cross section.



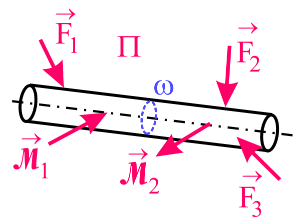
Example 106

10.3. Inner resultants in bars (VVÚ)

We solve an *elasticity problem* for a bar-type body loaded by a system of loads Π (volume, area and line distributed forces and/or isolated forces and couples). Displacements of the individual points of the body represent a visible demonstration of results of the loading process; these displacements are described mathematically by a vector field, i.e. by a set of displacements vectors $\vec{u}_A$. Stress and strain states in each of the points of the body are an inner demonstration of the loading process; these states are described by the mutually dependent tensors T_σ and T_ε in every point of the body. T_σ

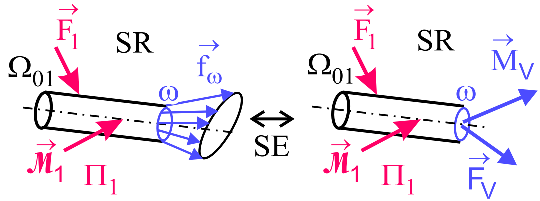
The solution to stress components is based on the equations of static equilibrium of an element of the body. T_ε

You divide the bar into two finite elements Ω_{01} and Ω_{02} by a cross section ω. The static equilibrium of the element Ω_{01} is ensured by the inner forces which have generally a character of area forces distributed continuously across the cross section ω; we introduced the quantity called *general stress* $\vec{f}_\omega$ to express these forces. (An analogical static equilibrium holds for element Ω_{02}; if the equilibrium conditions are satisfied for the element Ω_{01}, they will be automatically satisfied also for the element Ω_{02}.)



As the number of the independent equilibrium equations cannot be higher than six, they are not sufficient for evaluation of stresses, which can show another magnitude and direction in each point of the cross section; the **problem** of calculation of stresses in the cross section is multifold statically indeterminate. To make the solution possible, we replace the general stresses in the cross section by a statically equivalent (SE) force and moment resultant in its centroid R.

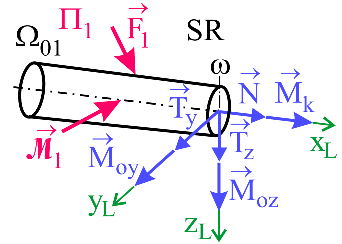
The resultants $\vec{F}_V$ and $\vec{M}_V$ are vectors, each of them having three components. This set of six components is called inner resultants (VVÚ – this abbreviation is based on the czech term) and they can be evaluated from the equations of static equilibrium (SR) of the element Ω_{01} or Ω_{02}, isolated as a free body; these equations express the equilibrium of outer forces Π_1 (or Π_2) and inner resultants $\Pi_V = \left\{\vec{F}_V, \vec{M}_V\right\}$, which act on the element Ω_{01} (or Ω_{02}).



equivalence

Mastering of evaluation of inner resultants is **necessary** to manage the elasticity theory of bars. The inner resultants are assistant quantities describing the loading of the bar and enabling us to find dangerous points (i.e. points with the lowest safety factor value) on the centreline of the bar in advance.

The procedure of definition of components of inner resultants is as follows:



We decompose the force and moment inner resultants into the directions of axes of the local coordinate system:

$$\vec{F}_V = \vec{F}_{Vx} + \vec{F}_{Vy} + \vec{F}_{Vz} = N\vec{i} + T_y\vec{j} + T_z\vec{k}$$

$$\vec{M}_V = \vec{M}_{Vx} + \vec{M}_{Vy} + \vec{M}_{Vz} = M_k\vec{i} + M_{oy}\vec{j} + M_{oz}\vec{k}$$

Their coordinates are components of **inner resultants** in the point R of the centreline (VVÚ).

$$\text{VVÚ} = \{N, T_y, T_z, M_k, M_{oy}, M_{oz}\}$$

The components of the inner resultants in a point of the centreline can be evaluated from the equations of static equilibrium of an element isolated as a free body. equations SR

The origin of the **local coordinate system** is in the centroid of the cross section in which the inner resultants are evaluated. The x_L-axis is identical with the centreline of the bar in the case of a straight bar (if the bar is curved then it is tangential to the curvilinear centreline), the axes y_L and z_L lie in the cross section and all the three axes create a Cartesian coordinate system. axes

Specific terms and notations are used for components of inner resultants:

N	-	normal force	⟨	orientation of outer normal - tension orientation of inner normal - compression
T_y, T_z	-	shear forces	-	shear loading of the bar
M_k	-	torsion moment (torque)	-	torsion loading of the bar
M_{oy}, M_{oz}	-	bending moments	-	flection loading of the bar

tension

torsion

flection

Inner resultants (VVÚ) are components of the force and moment resultants of inner forces in the centroid of the cross section, which together with the system of loads create an equilibrium system of forces acting on the element of the bar.

centroid

The distribution of inner resultants $\vec{F}_V, \vec{M}_V$ represents functions describing the distribution of their components along the centreline; these functions are determined by the shape of the centreline and by the loads. The centreline is deformed under load, therefore the resultants $\vec{F}_V$ and $\vec{M}_V$ can change in any point of the centreline during the loading process. Therefore the **deformed shape** of the centreline should be taken into account in evaluation of inner resultants using the second order theory of elasticity. If the changes of $\vec{F}_V$ and $\vec{M}_V$ in the consequence of centreline deformations are not substantial, these changes can be neglected and the resultants $\vec{F}_V$ and $\vec{M}_V$ can be evaluated with respect to the **undeformed** centreline, i.e. to the original shape (elasticity theory of the first order). In this case we must evaluate the deformation and to judge whether this deformation cannot change the inner resultants substantially. If they are changed substantially, the calculation is wrong and it should be repeated using the deformed shape of the centreline. This course, however, will deal with the elasticity theory of the first order (except for buckling of bars); then the deformation of the centreline does not influence the stress state substantially and the element can be isolated as a free body in its undeformed state.

loading

From the viewpoint of non-zero components of inner resultants, bar loading can be divided as follows:

- **onefold** loading of a bar – there is only one non-zero component of inner resultants $\vec{N}, \vec{T}, \vec{M}_o, \vec{M}_k$ in **all** points of the bar centreline. This loading is called tension ($N > 0$), compression ($N < 0$), flection ($M_o \neq 0$), torsion ($M_k \neq 0$) or shear ($T \neq 0$);
- **combined** loading of a bar – if **more than** one component of inner resultants $\vec{N}, \vec{T}, \vec{M}_o, \vec{M}_k$ are non-zero at least in one point of the centreline.

Notice that there are also two ways of expressing the inner resultants:

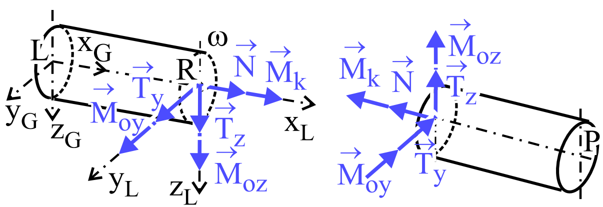
a) **Inner resultants in a point of the centreline:** a certain value of each of the components, it is necessary to calculate local characteristics (e.g. stresses).

b) **Inner resultants of the bar:** a function describing the distribution of each of the components along the length of the centreline, it is needed for calculation of global characteristics (e.g. displacements) and to find the dangerous points, as well.

Sign convention:

We introduce the following convention concerning signs of inner resultants (sometimes also other conventions are used in literature, the convention used should be always presented):

The quantities $N, T_y, T_z, M_k, M_{oy}, M_{oz}$ are supposed to be **positive** if their orientation is identical with the **positive** (negative) orientation of the local coordinate system axes for an element containing the original L (final P) point of the centreline.



Note: The different orientations of positive components of inner resultants on the left (containing the L point) and on the right (containing the P point) elements of the bar are introduced with the aim to satisfy the principle of action and reaction between the both elements. If we satisfy this convention, we obtain components of inner resultants with the same signs **no matter which** of both elements of the bar we have chosen for the solution.

10.4. Evaluation of inner resultants (VVÚ)

The objectives are as follows:

– to evaluate the components of inner resultants for a **general** point of the bar centreline,
– to represent the **distribution** of components of inner resultants along the centreline and to find the points where extreme values are reached,
– to calculate the **extreme** values of the individual components,
– to define parts of the centreline with the **same type of loading** using a set of non-zero components of the inner resultants.

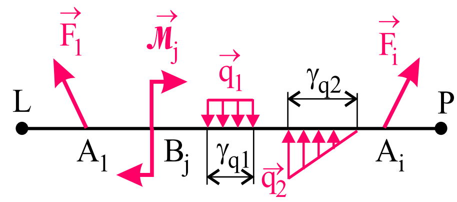
10.4.1. Approaches to evaluation of distribution of inner resultants (VVÚ)

In evaluation of distribution of inner resultants along the centreline, the following two approaches are used:

a) **Integral approach** – it is based on the formulation and solution of the equations of static equilibrium for a finite element of the bar.
b) **Differential approach** - it is based on the formulation and solution of the equations of static equilibrium for an (onefold) infinitesimal element of the bar.

Let's have a free straight bar the centreline of which is determined (in the global coordinate system) by the original L (left-handed) and final P (right-handed) points. The bar is loaded by a given general system of loads Π:

- isolated forces $\vec{F}_i$ [N] in the points A_i of the centreline, $i = 1 \div n$,
- isolated couples $\vec{M}_j$ [Nm] in the points B_j of the centreline, $j = 1 \div m$,
- line forces given by the load intensity $\vec{q}(l)$ $[\mathrm{Nm}^{-1}]$ per unit length of the centreline γ or its part γ_q; the points of the loaded part will be denoted by C.



Example 201

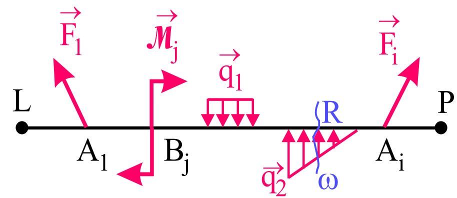
Dimensions of the bar cross sections need not to be defined for evaluation of components of inner resultants!

Example 202

a) Integral approach

is based on the definition of the components of inner resultants, which are evaluated from the equations of static equilibrium of a finite element in the following way:

1. We introduce a section ω containing the point R; this section divides the bar into two elements: Ω_L (containing the point L) and Ω_P (containing the point P).

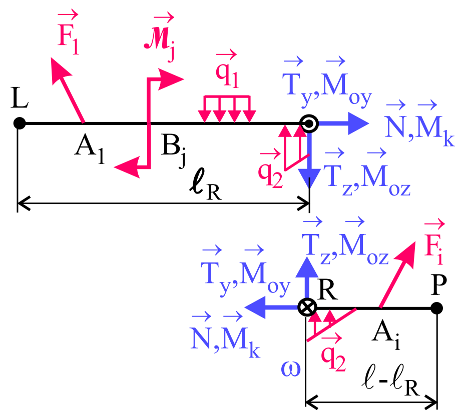


2. We evaluate the components of the inner resultants from the equations of static equilibrium of **one** of these elements, no matter which of them we choose. Usually we choose the element being easier for the solution.

3. The chosen element (denoted asΩ_R) has the centreline length l_R and it is loaded by a system of loads Π_R. We introduce the components of inner resultants ($\vec{F}_\mathrm{V}$, $\vec{M}_\mathrm{V}$) in the gravity centre of the cross section with the positive orientation according to the above sign convention. The bar is in static equilibrium, therefore the element Ω_R must satisfy the conditions of static equilibrium as well.

equilibrium

convention



– force condition: $\sum\limits_{l_R} \vec{F}_i + \int\limits_0^{l_R} \vec{q}_i(l)\mathrm{d}l + \vec{F}_V = \vec{0}$

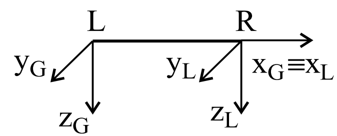
– moment condition:

$$\sum_{l_R}(\vec{RA}_i \times \vec{F}_i) + \sum_{l_R}(\vec{M}_j) + \int_0^{l_R}(\vec{RC} \times \vec{q})\mathrm{d}l + \vec{M}_V = \vec{0}$$

It is evident from the above vector equations (there are sums and integrals in them) that

inner resultants in the point R of the bar loaded by a system of external loads equal the sum of inner resultants created by the individual external loads.

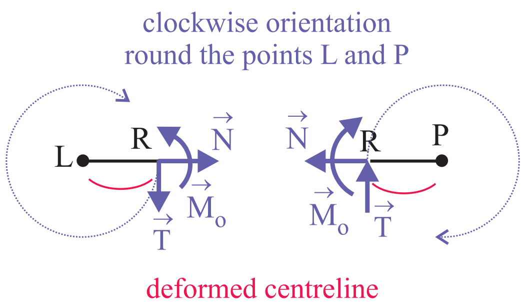
4. If we define the position of the point R of the bar centreline in the global coordinate system (x_R for a Cartesian, φ_R for a polar coordinate systems), we are able to evaluate any of the components of inner resultants as a function of the position of the point R along the bar centreline, i.e. to evaluate the **distribution of the inner resultants**.



local c.s.

Consequences of the sign convention:

- the positive normal force $\vec{N}$ is oriented outwards the section,
- the positive shear force $\vec{T}$ is oriented clockwise around the points L or P,
- the positive bending moment $\vec{M}_o$ deforms the bar in a convex shape (the red curve in the figure - the center of curvature is up).



The rules for signs of T and M_o can be unambiguously used only in a plane (2D) problem, at a straight horizontal bar.

5. Where the section should be introduced to obtain the distribution of the inner resultants?
   For evaluation of inner resultants, the bar must be described by its centreline being a continuous and smooth curve and a system of loads acting on the centreline need to be defined. The distribution of all the components of inner resultants along the centreline can be expressed in the form of functions with a finite number of discontinuity points along the centreline. These points represent borders of intervals and in each of these intervals one section must be introduced. Components of inner resultants are namely expressed in the form of functions, which can be discontinuous or have a discontinuous derivative on the borders of intervals.

   bar assumptions

6. We determine the distributions of the functions describing dependencies of the individual components of inner resultants on the section position (you have learned it in mathematics) and we find the position of extremes (in addition to the borders of intervals they can be in any points where the first derivative of the function equals zero) in an analytical or graphical way. In these so called dangerous points we calculate function values of the non-zero components of inner resultants.

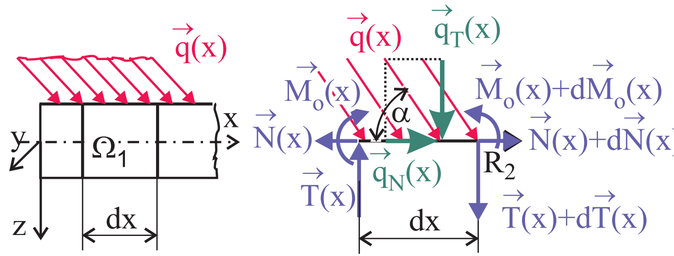
   dangerous point

b) Differential approach

It is based on differential dependencies between the bar loads and the components of inner resultants. These dependencies (called **Schwedler's theorems**) can be derived for a bar with a general centreline shape under a general load. Here, however, we derive only the differential relations valid for a *straight bar loaded by a general non-constant distributed in-plane load* $\vec{q}(x)$. We cut a onefold elementary element Ω_1 from the bar using two adjacent cross sections; the element has an infinitesimal length dx.

We decompose the continuous load $\vec{q}(x)$ in the normal and tangential directions of the cross section $(\vec{q}_T(x), \vec{q}_N(x))$:

$q_N(x) = q(x)\cos\alpha,\ q_T(x) = q(x)\sin\alpha$



Then we introduce the components of inner resultants; the differences between their magnitudes in both sections equal to the elementary increments $\mathrm{d}N, \mathrm{d}T, \mathrm{d}M_o$. We formulate the applicable conditions of static equilibrium; in this formulation, the distributed load $\vec{q}$ of the element can be considered constant (in magnitude as well as in direction) because of the elementary length of the element:

$$\sum F_x = 0: \quad N(x) + \mathrm{d}N(x) - N(x) + q_N(x)\mathrm{d}x = 0$$

$$\sum F_z = 0: \quad T(x) + \mathrm{d}T(x) - T(x) + q_T(x)\mathrm{d}x = 0$$

$$\sum M_{R_2} = 0: \quad M_o(x) + \mathrm{d}M_o(x) - M_o(x) - T(x)\mathrm{d}x + q_T(x)\mathrm{d}x\frac{\mathrm{d}x}{2} = 0$$

If we neglect the differential of the 2nd order against the other terms (differentials of the 1st order) in the last equation, we obtain relations denoted as **Schwedler's theorems**:

$$\frac{\mathrm{d}N(x)}{\mathrm{d}x} = -q_N(x), \qquad \frac{\mathrm{d}T(x)}{\mathrm{d}x} = -q_T(x), \qquad \frac{\mathrm{d}M_o(x)}{\mathrm{d}x} = T(x).$$

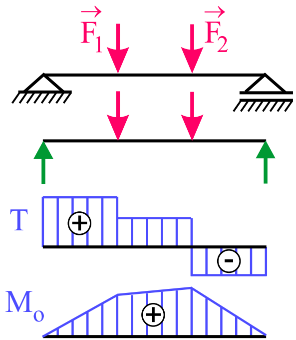
Let's analyse these relations from the viewpoint of the meaning of derivatives:

– The magnitude of the distributed load determines the direction of the tangent line to the curve representing the functional dependency of $T(x)$ in the point in question of the centreline.
– The magnitude of the shear force $T(x)$ in a certain point of the centreline equals to the direction of the tangent line to the curve describing the distribution of the bending moment $M_o(x)$.

If we namely know the distributed load, the character of distribution of components of inner resultants is unambiguously determined by this fact. To calculate the particular values, the following helping rules can be used.

10.4.2. Helping rules for evaluation of distribution of inner resultants

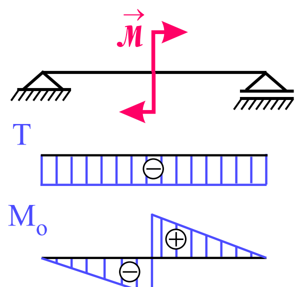
1. A step in the distribution (i.e. the direction of the tangential line $\rightarrow$ infinity) of $N(x)$ or $T(x)$ can be only there where an isolated force of the corresponding direction acts ($\vec{q} \rightarrow \infty$).
   $T > 0$, if the transverse force acts upwards on the left-hand side of the cross section.



2. In a point where there is a step in the $T(x)$ distribution (different values on the left and right sides of the section), a turning point must occur in the bending moment distribution $M_o(x)$ (different directions on the left and right sides).

$$\left(\text{Schwedler's theorem:} \qquad \frac{\mathrm{d}M_o(x)}{\mathrm{d}x} = T(x)\right).$$

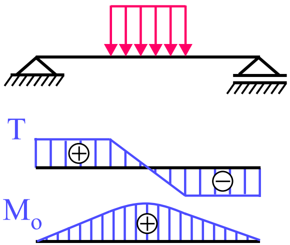
3. A step in the $M_o(x)$ distribution occurs if and only if there is an external couple acting in the point in question.
   $M_o > 0$, if the center of curvature of the deformed centreline is on its upper side.



4. If the bar is loaded only by isolated forces and couples (there are no distributed loads), the distributions of $N(x)$ and $T(x)$ are constant and the representation of $M_o(x)$ is given by straight lines (no curves), which can be different for each of the intervals (again a consequence of Schwedler's theorem).

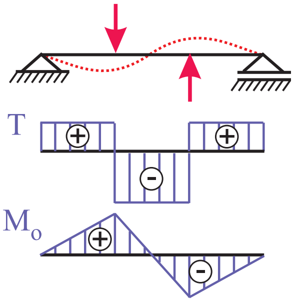
5. Where the representation of $T(x)$ is crossing the zero-line, there is an extreme in $M_o(x)$ function.

$$\left(\frac{\mathrm{d}M_o(x)}{\mathrm{d}x} = T(x) = 0 \quad \rightarrow \quad \text{extreme}\right)$$



6. In the bar cross section where the shear force is positive (negative), the function for $M_o(x)$ is increasing (decreasing) (a consequence of Schwedler's theorem: $\frac{\mathrm{d}M_o(x)}{\mathrm{d}x} = T(x)$).

7. According to the introduced convention it holds $M_o > 0$ for a convex deformed bar centreline. The convex and concave parts of the deformed bar centreline are joined together in the inflection point where, consequently, the bending moment must equal zero.



8. In the end of the bar, all the components of inner resultants must reach zero values, if there is no corresponding component of isolated external load acting in this final point (this would create a step in the distribution according to the par. 1 or 3).

9. For the evaluation of the distribution of components of inner resultants, advantage can be taken of the symmetry and antisymmetry of the bar.

If the bar is **symmetric** from the viewpoint of geometry and, from the viewpoint of **external loads** (incl. reactions in supports), it is

symmetric, then there is a

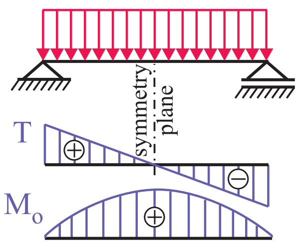
- zero shear force,
- extreme bending moment,
- zero torsion moment,

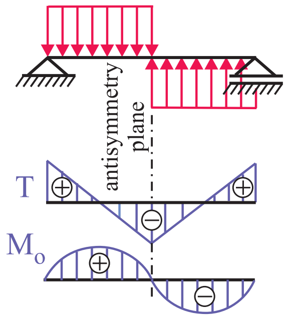
in the symmetry plane.

antisymmetric, then there is a

- zero normal force,
- extreme shear force,
- zero bending moment.

in the antisymmetry plane.

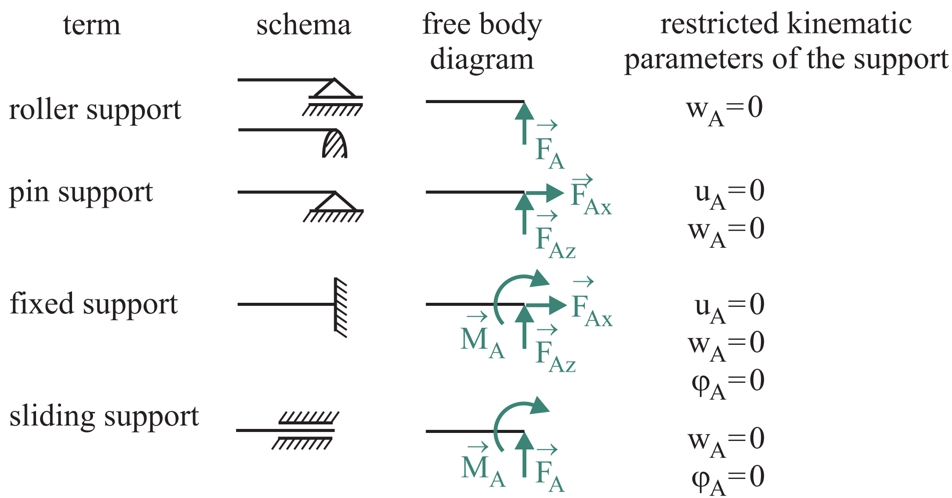




10.4.3. Opened supported bars

Equations of compatibility

A support is from the kinematic viewpoint described by a **set of kinematic parameters of the support** (displacements, rotations); in a 3D space, it is the set $D_3 = \{u, v, w, \varphi_x, \varphi_y, \varphi_z\}$, in plane $D_2 = \{u, w, \varphi\}$. If any of the kinematic parameters from the set D_i is restricted by the support, then the corresponding force parameter of support (component of the reaction force or couple) from the set $S = \{F_x, F_y, F_z, M_x, M_{oy}, M_{oz}\}$ is non-zero. Let's remember basic types of in-plane supports: support



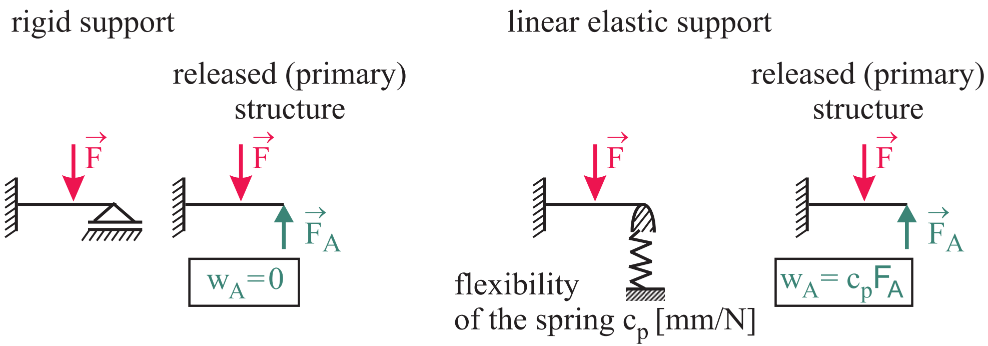
All these supports are **rigid** – restricted kinematic parameters are independent of the load and equal zero.

The **linear elastic supports** are characterised by linear relations between the correspon- ding components of force and kinematic parameters of the supports (e.g. $u = \mathtt{c} \,.\, F_x$). They are more realistic and should be used, if the deformations of the base are not neg- ligible against the deformations of the bar to be solved. In practice, it is often easier to determinate the stiffness (flexibility) of supports in an experimental way.

Problem 407

The equation determining the magnitude of a deformation (kinematic) parameter restric- ted by the support is denoted as **equation of compatibility** or **support deformation condition**. These equations are used in solutions to statically indeterminate bars.

Examples of compatibility equations for rigid and flexible supports:



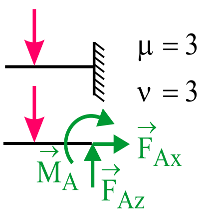
Bar supported in n points of the centreline

Let's have a bar under conditions of simple loading, being supported (joined with the base) in n points of its centreline. To solve the support reactions, we isolate the bar as a free body; it means that we remove the supports and replace them by the corresponding components of reaction forces or couples. When you dealt with statical analyses in statics, μ was introduced as a symbol denoting the number of unknown independent force parameters (components of unknown forces and couples) and ν as a symbol denoting the number of applicable equations of static equilibrium (it depends on the type of the force system). The decision, if the problem is statically determinate (soluble in statics), was based on the relation between these two numerical values. In opposite to dynamics, in stress analysis the only unknown parameters are reaction forces and couples, therefore we can use the term **statically determinate** (indeterminate) **bearing** of the bar.

statical analyses

The static analysis can result in the following conclusions:

a) $\boxed{\nu = \mu}$

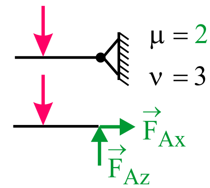


deformation characteristics

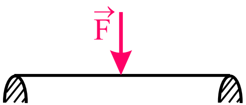
- the bearing is statically determinate,
- the unknown independent parameters of reaction resultants can be calculated from the applicable equations of static equilibrium.

The bearing of the bar is stable from the viewpoint of the whole body movement but with the possibility of free deformation (no deformation parameter is restricted).

b) $\boxed{\mu < \nu}$

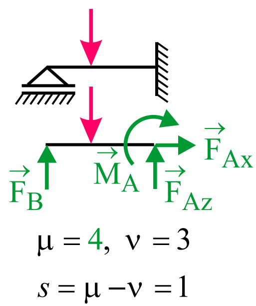


The bearing of the bar does not fully restrict its whole body movement. These problems are solved in dynamics and as late as dynamics has been solved, stress analysis can eventually be carried out.



Dynamics is not needed for the solution in the case that the movement of the body is otherwise possible but it will not happen under the given load. The problem is statically determinate ($\mu = \nu$) although the bearing does not ensure the immobility of the body.

c) $\boxed{\mu > \nu}$



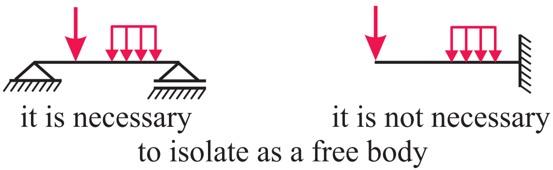
1. the bearing is statically indeterminate, the degree of static indeterminacy $s = \mu - \nu$,
2. the number of the independent unknown parameters of reaction resultants is higher than the number of applicable conditions of static equilibrium,
3. it is necessary to formulate s equations of compatibility (support deformation conditions) in addition to ν applicable conditions of static equilibrium, to solve the independent unknown parameters of reaction resultants.

The bearing of the bar as a whole is immovable and, moreover, restricted in deformation. We create the **released (primary) structure** to solve the problem.

Opened bars with static determinate bearing – the procedure of solution

We **isolate** the bar as a free body, that means we replace all the supports with reaction resultants; these resultants can be calculated from the equations of static equilibrium and the bar can then be solved like a free bar.

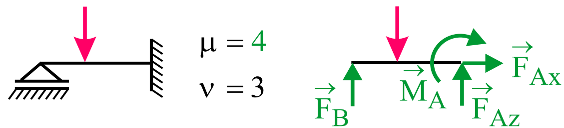
In some cases (a cantilever beam with one free end), the free body diagram is not needed to evaluate the inner resultants.



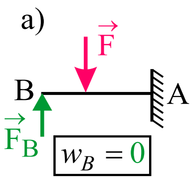
Opened bars with static indeterminate bearing – the procedure of solution

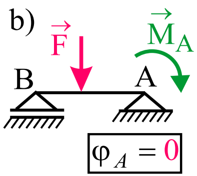
1. We **isolate the bar** as a free body, that means we replace all the supports by reaction resultants, and we formulate the applicable equations of static equilibrium.

   For example:
   $\sum F_x = 0:$ $\sum F_z = 0:$ $\sum M_A = 0:$



2. We create a **released (primary) structure**, i.e. we create a corresponding statically determinate structure (that means a structure with a fixed position in space but without any restricted deformation parameter) and formulate the compatibility equations (support deformation conditions); these conditions must be satisfied by the released supports (the reactions in these supports are called **statical redundances**).





The form of the equation of compatibility is unambiguously determined by the chosen released structure.

The **released (primary) structure** is a statically determinate structure created from the structure to be solved by replacing some supports by reaction resultants (statical redundances); together with creation of this released structure, equations of compatibility should be formulated to ensure the deformation identical with that of the original statically indeterminate structure. The formulation of these equations is the objective of this procedure.

Equation of compatibility (support deformation condition) is an equation expressing the restriction of deformation in the location of the released support. It is the very „missing“ equation needed for solving the unknown parameters of reaction resultants.

The equations of compatibility can be

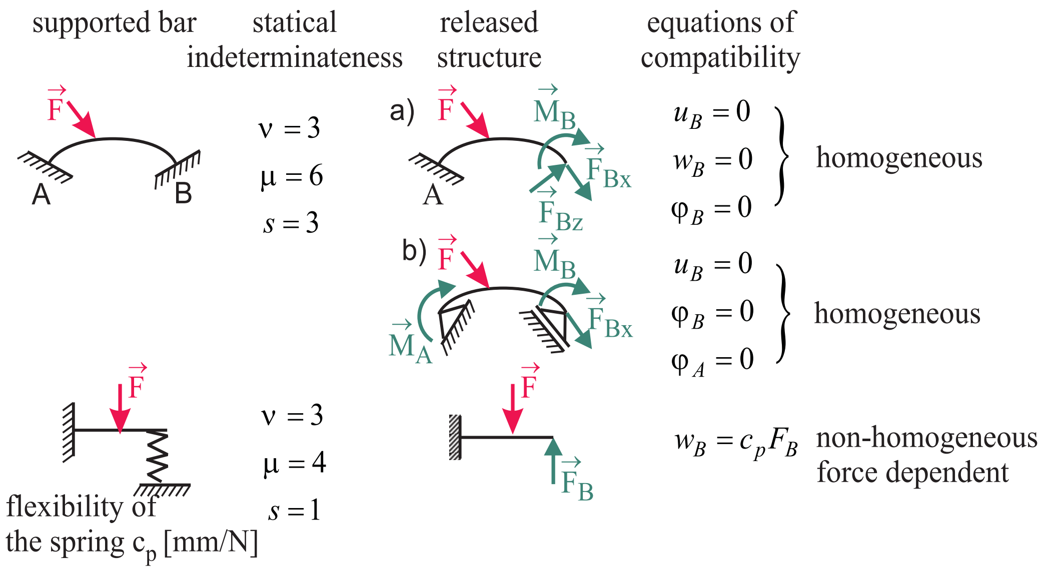
1. homogeneous (zero on the right-hand side of the equation) – at rigid supports, Example 414

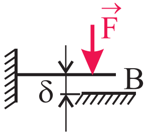
2. non-homogeneous – at flexible supports, bars with production inaccuracies, temperature changes, Example 418

Example 417

Example 419

3. circumstantial supports – at circumstantially acting supports. Example 437



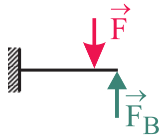


a) pro $w_B \leq \delta$ circumstantially statically determinate

$\nu = 3,\ \mu = 3,\ s = 0$

b) pro $w_B > \delta$ circumstantially onefold statically indeterminate

$\nu = 3,\ \mu = 4,\ s = 1$



non-homogeneous circumstantial

$\boxed{w_B = -\delta}$

The negative sign results from using the Castigliano's theorem where it means the displacement oriented against the direction of acting of the force $\vec{F}_B$. Castigliano's theorem

10.4.4. Closed bars - frames

Evaluation of inner resultants is always a statically indeterminate problem at closed bars. If we cut a closed bar with a single section, the bar will not be divided into parts (elements) but it only becomes opened. There are therefore no applicable conditions of static equilibrium to evaluate the inner resultants. Moreover, the problem can show an **outer** statical **indeterminacy** as well, in dependence on the character of the bearing (supports joining the body with the base). For the solution it is necessary to transform the closed bar into an opened one by suitably introduced sections and to formulate equations of compatibility. However, we will not deal with the solutions to closed bars in more detail in this course.

10.4.5. Algorithm of evaluation of inner resultants

1. **Classification of the bar**
   - An opened straight statically determinate bar – isolation as a free body and calculation of reactions in supports (if they are needed).
   - An opened straight statically indeterminate bar – only a qualitative solution is possible (for a quantitative solution it is necessary to create a released structure and to formulate the equations of compatibility to complete the set of equations of static equilibrium).
   - An opened curved bar – the solution is similar to the straight bar; however, the force components of inner resultants are not constant even if there is no distributed load.
   - A closed bar – is always statically indeterminate from the viewpoint of evaluation of inner resultants (inner statical indeterminacy), it can be moreover outer statically determinate or indeterminate. Evaluation of the distribution of inner resultants is always a relatively complex problem requiring to complete the equations of static equilibrium by the needed number of compatibility equations.
2. **Isolation of the bar** as a free body – formulation of the conditions of static equilibrium and calculation of reactions in supports (if the bar to be solved is not a cantilever beam and the problem is statically determinate).
3. **Division** of the bar into intervals; this must be realized in all points where:
   - isolated outer loads (forces or couples, incl. reactions in supports) act;
   - there is a change in the character of a distributed load;
   - where the direction of the centreline (turning point) or its curvature changes. Example 203
4. Decision on the **further procedure** (it need not be the same for all the intervals)

- integral approach (formulation of the equations of static equilibrium for an element of the bar) should be used in those cases, when the problem is relatively complex but statically determinate. integral approach
- differential approach (application of Schwedler's theorems) can be used if the task is relatively easy; however, it must be used always at statically indeterminate bars (a qualitative solution). differential approach

A) differential approach

5. We evaluate the distribution of the force components of inner resultants or of the torsion moment from the given distribution of continuous load, using the 1st Schwedler's theorem and other rules (10.4.2 Helping rules for evaluation of distribution of inner resultants in straight bars).
6. We evaluate the **distribution of the bending moment** from the distribution of the shear force using the 2nd Schwedler's theorem.
7. We estimate the dangerous sections from the distribution of the components of inner resultants (locations of local extremes) and we calculate the magnitudes of components of inner resultants in these points of the centreline (if the problem is not statically indeterminate).

B) integral approach - we carry out the following steps in each of the intervals of the bar:

5. We **isolate an element** of the bar as a free body using a section going through a general point of the interval in question.
6. We evaluate all the components of inner resultants from the equations of static equilibrium of the element as functions of the position of the section.
7. We evaluate the functional dependencies of components of inner resultants from the viewpoint of local extremes and define the position of these extremes.
8. We draw the distribution of the components of the inner resultants, define the dangerous cross sections and calculate the magnitudes of components of the inner resultants in these sections.

Schwedler's theorem

Example 238

Note: *at non-prismatic bars (with the cross section varying along the bar centreline), more dangerous sections can exist because of the local reduction of the bar cross section. This will not be taken into account now but when solving stresses and deformations for the particular types of loadings.*

10.5. Quadratic moments - examples and problems

Examples

Problem 101 Problem 102 Problem 103 Problem 105 Problem 106

Problems

Problem 104 Problem 107 Problem 108 Problem 109 Problem 110

10.6. Inner resultants - examples and problems

Examples

Problem 201	Problem 202	Problem 203	Problem 217	Problem 238

Problems

Problem 204	Problem 205	Problem 206	Problem 207	Problem 208
Problem 209	Problem 210	Problem 211	Problem 212	Problem 213
Problem 214	Problem 215	Problem 216	Problem 218	Problem 219
Problem 220	Problem 221	Problem 222	Problem 223	Problem 224
Problem 225	Problem 226	Problem 227	Problem 228	Problem 229
Problem 230	Problem 231	Problem 232	Problem 233	Problem 234
Problem 235	Problem 236	Problem 237		