13. Simple flection

13.1. Definition

Simple flection is loading of a straight prismatic bar, if
- bar assumptions are satisfied,
- cross sections rotate around an axis lying in the cross section and, consequently, deform,
- the only non-zero components of the inner resultants are bending moments $\vec{M}_{oy}, \vec{M}_{oz}$,
- deformations of the bar are not significant from the viewpoint of element equilibrium.

bar assumptions

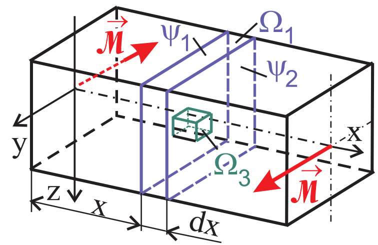
Note: It results from the Schwedler's theorem $T = \mathrm{d}M_o/\mathrm{d}x$ that $\vec{M}_o$ must be constant if the shear force $T = 0$. This is exactly satisfied only at bars loaded by couples.

Since at the simple flection two of the components of inner resultants are non-zero ($\vec{M}_{oy}, \vec{M}_{oz}$), the solution is more complex than in the case of other types of loading. This type of flection is called **general flection** (sometimes also inclined or 3D flection).

To simplify the derivation, all the relations will be derived first for the so called **basic flection** (only one component of bending moment is non-zero), in particular for $M_{oy} \neq 0, M_{oz} = 0$.

13.2. Geometrical relations

We isolate a onefold elementary element Ω_1 as a free body from the bar and then again a threefold elementary one Ω_3 from it. The element Ω_1 deforms in such a way that the adjacent sections ψ_1 and ψ_2:

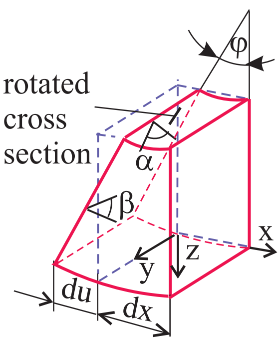


– rotate around an axis lying in the cross section, and the original length dx of the element Ω_3 changes by an increment du,

– remain perpendicular to the deformed bar centreline, so that the right angles α and β of the element Ω_1 and Ω_3 do not change.

bar assumptions

Since the cross section remains planar, according the bar assumptions, also after its rotation and it rotates around a straight line parallel to the y-axis under conditions of basic flection ($M_{oy} = M_o \neq 0$), the displacements du are independent from y coordinate and they can be described by an linear equation (an equation of a straight line in (x, z) plane) $\mathrm{d}u(z) = a_1 + b_1 z$. The following components of the strain tensor correspond to this deformation:



– length strain in the direction of the bar centreline,

$$\varepsilon_x(z) = \frac{\mathrm{d}u(z)}{\mathrm{d}x} = a + bz,$$

– zero angular strains $\gamma_{xy} = \gamma_{xz} = 0$.

In consequence of the transversal contraction, transversal strains $\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x$ come into existence, different in magnitude in each point of the bar.

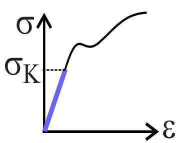
strain

In the case of simple flection, distribution of the length strains is linear throughout the cross section and the angular strains equal zero.

A general triaxial strain state comes thus into existence in any point of the bar; this strain state is described by the strain tensor in the form $T_\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$. In contrast to the simple tension, the strain state is non-uniform throughout the cross section, the values are different in each of its points. tensor strain

13.3. Stress distribution throughout the cross section

If the material is Hookean (homogeneous, linear elastic), the distribution of the normal stress σ_x is also linear, similar to strain ε_x: Hook's law



$$\sigma_x(z) = E\varepsilon_x(z) = E(a + bz).$$

Shear stress is determined by the relation: $\tau = \dfrac{E}{2(1+\mu)}\gamma = G\gamma.$

Since $\gamma_{xy} = \gamma_{xz} = 0$, it holds also $\tau_{xy} = \tau_{xz} = 0$.

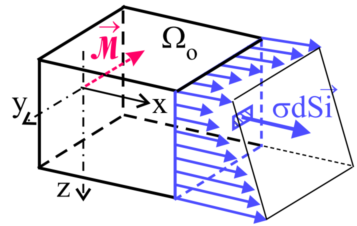
The other components of the stress tensor $(\sigma_y, \sigma_z, \tau_{yz})$ equal zero because of the bar assumptions. Therefore, the only non-zero stress component is the normal stress σ_x with a **linear** distribution throughout the cross section. bar assumptions

In the case of simple flection, a **uniaxial stress state** comes into existence in the points of a bar but, in contrast to the simple tension, it **is not uniform**.

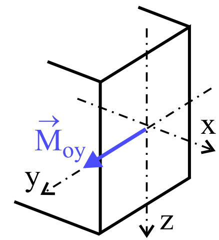
13.4. Dependence between inner resultants and stresses

static equivalence

The relation for stress $\sigma(z)$ can be derived from the equations of static equivalence between the system of inner elementary plane forces in the cross section $\sigma \mathrm{d}S\vec{i}$ and their resultant $\vec{M}_{oy}$ in the cross section ψ of the element Ω_0; we formulate this equations in the local coordinate system acc. the figure. There are three applicable conditions of static equivalence for a system of parallel forces in a 3D space:



static equations

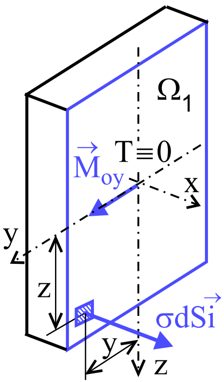


$$\iint_{\psi} \sigma \mathrm{d}S = 0, \quad M_{oy} = \iint_{\psi} z\, \sigma \mathrm{d}S, \quad M_{oz} = -\iint_{\psi} y\, \sigma \mathrm{d}S \; = 0.$$

We substitute $\sigma = E(a + bz)$:

stress

central c.s.



$$E \iint_\psi (a + bz)\mathrm{d}S = 0 \quad \Rightarrow \quad a \iint_\psi \mathrm{d}S \; + b \iint_\psi z\mathrm{d}S = 0 \quad \Rightarrow \quad a = 0,$$

because $\iint_\psi z\mathrm{d}S = U_y = 0$ in a centroidal coordinate system

$$M_{oy} = E \iint_\psi (a + bz)z\mathrm{d}S = E(a \iint_\psi z\mathrm{d}S \; + b \iint_\psi z^2\mathrm{d}S) \quad \Rightarrow \quad b = \frac{M_{oy}}{EJ_y}$$

By substituting a and b in the relation for the stress we obtain

$$\sigma = E(a + bz) = E\frac{M_{oy}}{EJ_y}z \quad \Rightarrow \quad \boxed{\sigma = \frac{M_{oy}}{J_y}z.}$$

However, the relation is valid if and only if the third applicable condition of static equivalence is satisfied; this is the case in a **principal centroidal coordinate system** only:

$$M_{oz} = -E \iint_\psi (a + bz)y\mathrm{d}S = -E\frac{M_{oy}}{EJ_y} \iint_\psi yz\mathrm{d}S = \frac{M_{oy}}{J_y}J_{yz} = 0 \quad \Rightarrow \quad J_{yz} = 0$$

principal
central c.s.

Note:

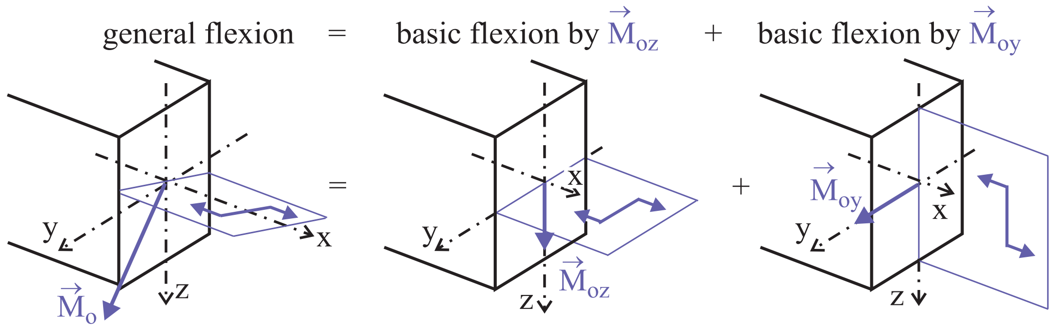
For the non-zero bending moment M_{oz}, a similar relation holds for stress:

$$\sigma = -\frac{M_{oz}}{J_z} y.$$

Since both of these stresses are parallel to the x axis, we can calculate the resulting stress for the **general (inclined) flection** by addition of them:

$$\sigma = \frac{M_{oy}}{J_y} z - \frac{M_{oz}}{J_z} y.$$

All these relations are valid in a principal centroidal coordinate system only. Therefore the basic flection comes into being just then if the line of action of the bending moment is identical with one of the principal centroidal axes of the cross section (e.g. with a symmetry axis).



13.5. Extreme stresses

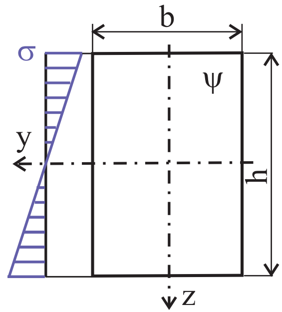
To simplify the description of the stress distribution in the cross section, we introduce first the so called **neutral axis**, what is a straight line with the following properties:

- it lies in the cross section plane and contains its centroid,
- it holds $\sigma = 0$ and, consequently, $\varepsilon = 0$ in all of its points,
- it divides the cross section into two parts, one of them having positive and the other one negative stresses.

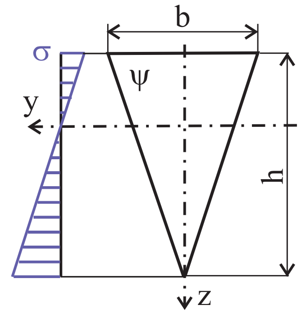
It is evident from the formula for stresses in basic flection $(M_{oy} \neq 0)$ that the neutral axis is identical with the y axis, and, at the same time, with the line of action of the bending moment. Since the stress distribution is linear, the extreme absolute values must be in points with maximum distance from this axis.

$$\sigma_{\max} = \frac{M_{oy}}{J_y} z_{\max}$$

Therefore dangerous points are the points with the maximum absolute value of z coordinate. The so called **section modulus** W_o [m^3] can be introduced for the basic flection; this modulus is defined as the ratio of the principal **centroidal** axial quadratic moment (related to the neutral axis y) and of the maximum distance of an outline point from the neutral axis y, $W_o = J_y/z_{\max}$). Then we obtain the following formula for the maximum stress:



$$z_{\max} = \frac{h}{2}$$

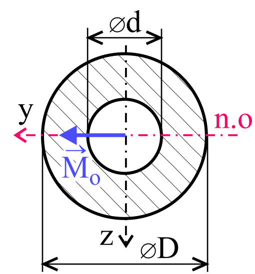


$$z_{\max} = \frac{2}{3} h$$

$$\sigma_{\max} = \frac{M_{oy}}{J_y} z_{\max} = \frac{M_o}{W_o}.$$

Warning! **Section** modulus is not additive. For instance for an annular section, it must be calculated by subtraction of axial quadratic moments, while the maximum distance $z_{\max} = D/2$ remains the same!

quadratic moment



$$W_o = \frac{J_y}{\frac{D}{2}} = \frac{\frac{\pi D^4}{64} - \frac{\pi d^4}{64}}{\frac{D}{2}} = \frac{\pi D^3}{32}\left[1 - \left(\frac{d}{D}\right)^4\right]$$

The evaluation of extreme stresses in general flection is much more complex.

13.6. Strain energy

Under assumptions of the linear theory of elasticity, all the deformation work is transformed into the reversible strain energy ($A = W$). The following relation was derived in chapter 11.6. for strain energy of a threefold **infinitesimal** element under conditions of uniaxial stress state strain energy

$$W_{\Omega_3} = A_{(\sigma \mathrm{d}S)} = \Lambda \mathrm{d}S\mathrm{d}x = \frac{1}{2}\frac{\sigma^2}{E}\mathrm{d}S\mathrm{d}x.$$

The strain energy of a onefold **infinitesimal** element$\Omega\ _1$ can be obtained by integration of the energy W_{Ω_3} (after substitution $\sigma(z) = \dfrac{M_{oy}}{J_y}z$ for stress) throughout the area ψ: stress

$$W_{\Omega_1} = \iint\limits_{\psi} \frac{1}{2}\frac{\sigma^2}{E}\mathrm{d}x\mathrm{d}S = \frac{1}{2E}\iint\limits_{\psi}\frac{M_{oy}^2}{J_y^2}z^2\mathrm{d}S\mathrm{d}x = \frac{M_{oy}^2}{2EJ_y}\mathrm{d}x,$$

because $\iint\limits_{\psi} z^2\mathrm{d}S = J_y$. The total strain energy accumulated in the bar of length l equals then the integral of strain energies of elements Ω_1 along the bar length

$$\boxed{W_= \int\limits_0^l W_{\Omega_1} = \int\limits_0^l \frac{M_{oy}^2}{2EJ_y}\mathrm{d}x.}$$

For the general flection ($M_{oy} \neq 0, M_{oz} \neq 0$), strain energy is given by superposition of contributions of both basic simple flections (from bending moment components $\vec{M}_{oy}, \vec{M}_{oz}$):

$$W = W_{M_{oy}} + W_{M_{oz}}.$$

The relations are valid only for the principal **centroidal** coordinate system ($J_{yz} = 0$)! principal c.s.

13.7. Description of centreline deformations

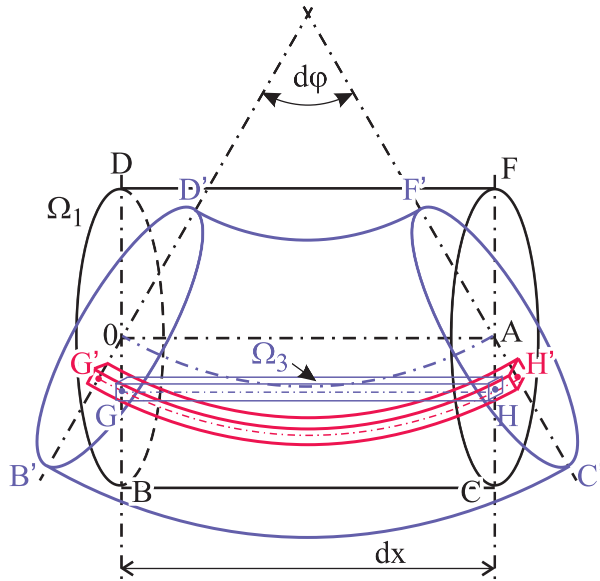
deformation characteristics

simple flection

If a straight prismatic beam is under flection, its centreline is bended and it creates the so called **deflection curve**. According to the bar assumptions, the cross sections remain planar and perpendicular to the deflection curve, so that displacements of any point of the bar can be calculated if we know **deflections** and **slopes** in individual points of the centreline (deflections are displacement components perpendicular to the centreline); therefore these characteristics are denoted as basic deformation characteristics of the bar under simple flection. They can be calculated from the differential equation of the deflection curve.

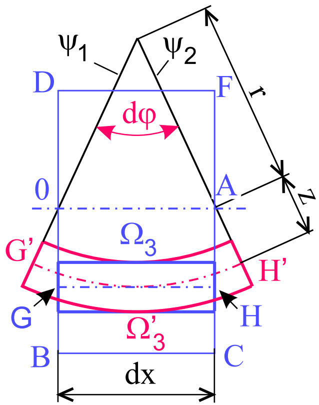
During the deformation of a onefold **infinitesimal** element Ω_1, the two adjacent sections rotate mutually by the angle $d\varphi$ around the neutral axis. Neutral axes in the individual cross sections create together a **neutral plane**; in all points of this plane, stresses and strains are zero. The length of a threefold **infinitesimal** element Ω_3 (given by the distance GH in the figure) changes into $\widehat{G'H'}$ by elongation and distortion of the element.

To derive the differential equation of the deflection curve, we assume again the **basic flection** with the line action of the bending moment identical with y axis $(\vec{M}_{oy} \neq 0, \vec{M}_{oz} = 0)$.



bar assumptions

neutral axis



strain

uniaxial stress state

stress

Hook's law

The elementΩ_3 with its centreline in the distance of z from the neutral axis had the length of $r\mathrm{d}\varphi$ before deformation (i.e. the same as the abscissa OA, the elongation of which is negligible) and the length of (r+z)dφ after deformation.
Then the length strain of the elementΩ_3 is

$$\varepsilon_{\Omega_3} = \frac{(r+z)\mathrm{d}\varphi - r\mathrm{d}\varphi}{r\mathrm{d}\varphi} = \frac{z}{r}$$

There is a uniaxial stress state under flection, and since we suppose the basic flection with $\vec{M}_{oy}$, it holds

$$\varepsilon_{\Omega_3} = \frac{\sigma}{E} = \frac{M_{oy}}{EJ_y} z.$$

By comparison $\frac{z}{r} = \frac{M_{oy}}{EJ_y} z \Rightarrow \frac{1}{r} = \frac{M_{oy}}{EJ_y}$ we can obtain the curvature $\frac{1}{r}$ of the deformed centreline, or the radius of curvature r of the centreline.

Note:
Analogically for the latter component of bending moment $\vec{M}_{oz}$, we can obtain the relation

basic flection

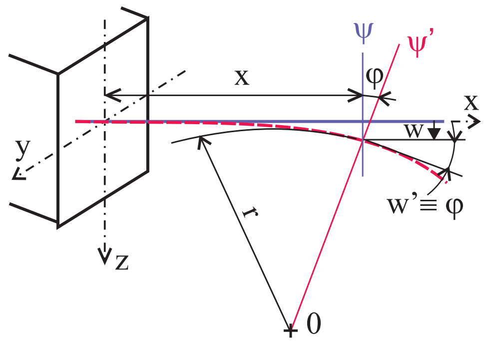
$$\frac{1}{r} = \frac{M_{oz}}{EJ_z}.$$

Since the term $\frac{M_{oy}(x)}{EJ_y(x)}$ is constant along the centreline (given by assumptions of the simple flection), the centreline is deformed into a circular arch (deflection curve). In practice, however, cases with $M_o(x) \neq$const. are much more frequent; consequently $\frac{1}{r} \neq$ const. and the deflection curve is a general 2D curve. (The influence of shear force that must occur if $M_o(x) \neq$ const. will be analysed in chapter 13.9.2.)

flection

influence T

In mathematics, the following relation was derived for the curvature of a planar curve representing the function $z = z(x)$



$$\frac{1}{r(x)} = \frac{\pm\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}}{[1 + (\frac{\mathrm{d}z}{\mathrm{d}x})^2]^{\frac{3}{2}}} = \frac{\pm w''}{(1 + w'^2)^{\frac{3}{2}}},$$

where w is the displacement of a point of the centreline in z direction (i.e. deflection). By comparison with the curvature derived above, we can obtain the differential equation of the deflection curve

$$\frac{\pm w''}{(1 + w'^2)^{\frac{3}{2}}} = \frac{M_{oy}}{EJ_y}.$$

It is a general non-linear differential equation of the 2nd order that can be solved in an analytical way in some special cases only.

Only small deformations are admissible at most of technical objects and structures; if the slopes are $\varphi < 0,1\,\mathrm{rad}$, it holds $w' = \ \mathrm{tg}\ \varphi \doteq \varphi$ and $w'^2 < 0,01$ can be neglected against 1. Under assumption of **small deformations**, we obtain a common linear differential equation of the 2nd order with a right-hand side; this equation can be solved by its direct integration:

$$\boxed{w'' = -\frac{M_{oy}}{EJ_y}.}$$

The negative sign in the equation occurs in consequence of the introduced sign conventions and orientation of coordinate axes.

13.8. Deformation of the cross section

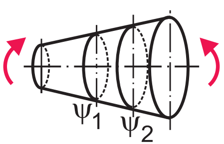
In consequence of the transversal contraction, the strains ε_y and ε_z are non-zero so that the cross sections change under deformation. The calculation of changes in cross section dimensions is more complex than in the case of the simple tension, because the strain state is non-uniform. These changes, however, are mostly not significant in practice. deformation

13.9. Fields of applicability of the theory of simple flection of bars

13.9.1. Influence of the cross section variability along the centreline

a) Continuously variable cross section

Let's have a straight bar with a cross section which changes continuously along the bar centreline; the bending moment is constant along the centreline and the principal axes of all the cross sections are parallel (the bar is not screw-shaped).



It was derived in chapter 11.10.1 that a shear stress occurs in the cross sections if $N \neq 0$. Similarly it can be derived for simple flection that a variability of the cross section magnitude along the centreline induces shear stresses in cross sections as well. derivation

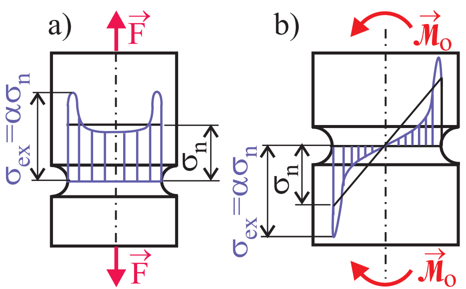
It holds here similarly to the simple tension that if the change in cross section magnitude is small, also the shear stress will be small in comparison with the normal stresses ($\tau \ll \sigma$) and this deviation from the bar assumptions can be neglected. The above relations of simple flection can then be used in calculations of stresses and deformation characteristics. bar assumptions

b) Stepwise changes in cross sections (notches)

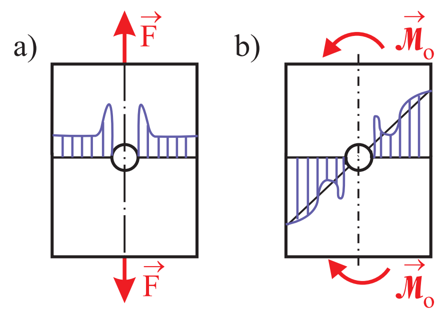
The location with the maximum stress value is called the root of the notch. The maximum stress value is calculated using the formula $\sigma_{\max} = \alpha\sigma_n$, where α is the stress concentration factor, σ_n is the nominal stress in the notch location, calculated using formulas of simple theory of elasticity.

notches
α graphs
stress
Problem 602

We can see the differences in stress distributions in the notch locations between two examples of bars loaded in a) tension and b) flection:



1. under flection, the stress concentration can occur simultaneously in both positive (tensile) and negative (compressive) parts of the cross section,
2. under flection, the notch location influences substantially the stress concentration (the character of stress concentration is different in dependence on the location of the notch in the cross section),
3. under flection, the maximum stress in the root of a notch near the neutral axis need not to exceed the nominal stress at the circumference, while under tension (because of the uniform stress state in the unnotched cross section) the stress in the root of the notch is always the highest in magnitude.

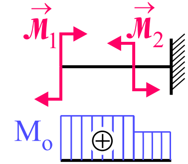


13.9.2. Variability of bending moment along the centreline

The assumptions of the simple flection can be satisfied only in the case of a bar loaded by isolated couples, if it holds:

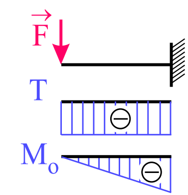
- shear force $T(x) = 0$,
- bending moment $M_o(x) = \mathcal{M} =$ const. in the individual intervals,

Then there are no shear stresses in cross sections.



In practice, bars loaded by isolated forces or distributed loads in transversal direction are much more frequent. In these bars, the shear force is non-zero and the bending moment is not constant; the term **beam** is generally used for this type of bars. The stress state in beams is of a more complex type:

- normal stresses σ occur in cross sections, **induced** by the bending moment $\vec{M}_o$,
- shear stresses τ occur in cross sections, **induced** by the shear force $\vec{T}$.



Transversal loads result always in shear stresses in cross sections of the beam.

The magnitude and distribution of the shear stresses in the cross sections with a general shape of the outline and general direction of the shear force cannot be solved but using methods of general theory of elasticity or finite element method.

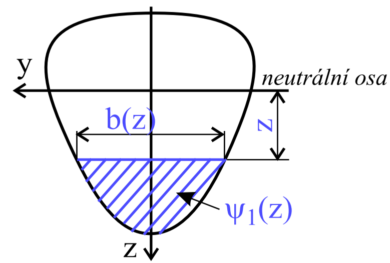
At the level of simple elasticity theory of bars, the following two cases can be solved:

1. cross section with one symmetry axis at least,
2. thin-walled cross sections - **I**, **U**, **T** profiles under assumption that
   - the beam is prismatic,
   - the beam surface is not loaded by shear loads.

The following formula (sometimes called Zhuravsky's or shear formula) is used for calculation of shear stresses.

$$\tau(x,z) = \frac{T(x)U_{y\psi 1}(z)}{b(z)J_y},$$

where $U_{y\psi 1}(z)$ is the first (static, linear) moment of the area $\psi_1(z)$ with respect to the neutral axis y.



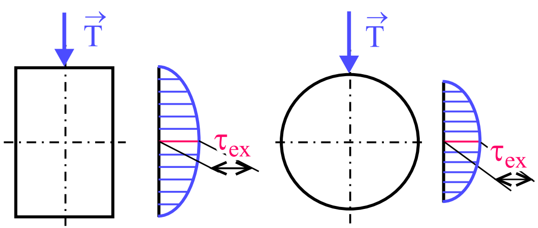
static moment

neutral axis

This formula was derived under assumption that the line of action of the shear force T_z is identical with the symmetry axis z of the cross section and the shear stresses are constant across its width ($\tau(y) = \text{const.}$). Using this formula, the following formulas for maximum shear stresses (in **centroid** of the cross section) can be derived:

a) in a rectangular cross section: $\tau_{\max} = \frac{3}{2}\frac{T}{S}$

b) in a circular cross section: $\tau_{\max} = \frac{4}{3}\frac{T}{S}$



Note: It is thus evident that the so called conventional shear stress $\tau_s = T/S$, used sometimes in practice, underestimates substantially the shear stress magnitude. Additionally, the assumptions of the shear formula are not satisfied in some of the profiles and the extreme shear stresses are even higher in reality.

To calculate the deformation parameters using Castigliano's theorem, the influence of the shear force should also be comprehend in the strain energy. The formula $\Lambda = \frac{\tau^2}{2G}$ was derived for the strain energy density **induced** by shear stresses. If the shear stress **induced** by the shear force $\vec{T}$ acts in the cross section the strain energy of a onefold **infinitesimal** element Ω_1 can be obtained by integration of this term throughout the cross section ψ Castigliano's theorem Λ

$$W_{\Omega_1} = \iint_{\psi} \frac{\tau^2}{2G} \mathrm{d}S\mathrm{d}x = \frac{1}{2G} \iint_{\psi} \frac{T^2 U^2_{y\psi 1}(z)}{b^2(z) J_y^2} \mathrm{d}x\mathrm{d}S.$$

After some manipulations (we add S in both numerator and denominator of the fraction and denote as β the term in brackets, which is function of cross section characteristics only and is constant for a certain cross section) we obtain:

$$W_{\Omega_1} = \frac{T^2}{2GS} \left[S \iint_{\psi} \frac{U^2_{y\psi 1}(z)}{b^2(z) J_y^2} \mathrm{d}S \right] \mathrm{d}x = \frac{\beta T^2}{2GS} \mathrm{d}x$$

It holds for a circular section $\beta = 32/27 = 1,185 \doteq 1,2$, for a rectangular $\beta = 1,2$.

The contribution of the shear force to the total strain energy of the beam with length l is thus: Example 627

$$\boxed{W_T = \int_0^l W_{\Omega_1} = \frac{\beta}{2G} \int_0^l \frac{T^2(x)}{S(x)} \mathrm{d}x.}$$

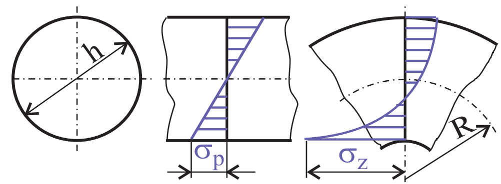
13.9.3. Beams with curved centreline

The normal stress distribution in the cross section of a beam with a curved planar centreline under basic flection is hyperbolic, in contrast to a straight beam where the distribution is linear; the neutral axis is shifted from the **centroid** towards the center of **curvature** of the beam. To compare the resulting stress values calculated using formulas for curved beams σ_z and for straight beams σ_p **(in a beam with** R being radius of curvature a h dimension of the cross section in the plane of the centreline), the dependency $\Delta\sigma(R/h)$ is repre-sented in the figure; $\Delta\sigma$ is calculated as follows:

basic flection

neutral axis

σ_p

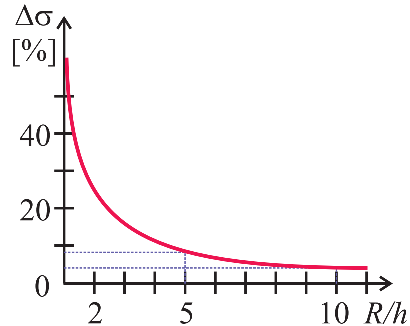


$$\Delta\sigma = \frac{\sigma_z - \sigma_p}{\sigma_z} \cdot 100\ \%.$$

The ratio R/h represents the inverse relative curvature of the beam, the value $\Delta\sigma$ is the relative deviation of σ_p from σ_z.

It is evident from the graph that the stresses in beams with low curvature

($h \ll R$, i.e. $\frac{R}{h} \gg 1$) can be calculated using formulas valid for straight beams; the relative error will be $\sim 4\%$ for $R/h = 10$ and $\sim 8\%$ for $R/h = 5$. For the beams with high curvature ($R/h < 5$) the stress distribution is **hyperbolic** (it cannot be replaced by a straight line), the extreme stress is higher and it must be calculated using a theory of beams with high curvature (which is not included in this bachelor course) or, more frequently today, using the finite element method.



13.10. Solving problems concerning simple flection of beams

13.10.1. Free beam

We derived the relations for stress, deformation parameters and strain energy valid for a bar under flection if the bar assumptions are satisfied. In this course, we restrict ourselves to basic flection in practical calculations; then the following simplified formulas are valid:

bar assumptions

$$\sigma(z) = \frac{M_{oy}}{J_y} z; \quad \sigma_{\max} = \frac{M_o}{W_o}; \quad w'' = -\frac{M_{oy}}{EJ_y}; \quad W = \int_0^l \frac{M_{oy}^2}{2EJ_y} \mathrm{d}x$$

$\sigma(z)$

$\sigma_{\max}$

w''

W

To judge the limit states of beam deformation, we need to know the deflections or slopes in some significant points of the beam centreline at least. There are many methods for evaluation of these deformation parameters of beams; we introduce two of them:

– integration of the differential equation of the beam's deflection curve (differential approach),
– Castigliano's theorem (integral approach).

13.10.2. Differential approach

The differential equation $w''(x) = -\dfrac{M_{oy}(x)}{EJ_y}$ can be solved by a direct integration; to be soluble, it must be completed by boundary conditions. If the $M_o(x)$ distribution along the beam centreline can be expressed by a single (continuous and smooth) functional dependency, one differential equation of the 2nd order is sufficient for the solution; two boundary conditions are needed to solve the integration constants.

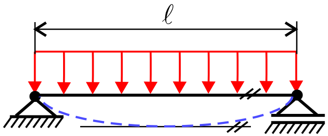
Problem 604

Problem 607

The boundary conditions can be described by:

a) support deformation conditions – known deflections or slopes in the locations where the beam is supported by the base,

b) symmetry of deformation,
for $x = \frac{l}{2} \rightarrow w' = \varphi = 0$ (the tangential line of the deflection curve is parallel to the x axis)
Thanks to this fact, there are two possibilities of how to express the boundary conditions at the beam in the figure:



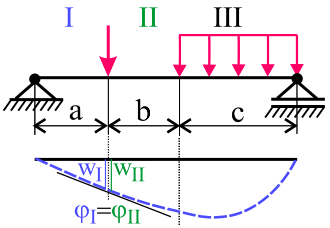
1. support conditions
   $x = 0 \quad w = 0$
   $x = l \quad w = 0$
2. symmetry of deformation
   $x = 0 \quad w = 0$
   $x = \frac{l}{2} \quad w' = 0$

c) geometrical bar assumptions (the bar centreline remains continuous and smooth during deformation). If the term M_{oy}/EJ_y is expressed by different functional dependencies in several intervals of the beam centreline, then we can formulate the conditions of continuity and smoothness of the deflection curve on the boundaries of the intervals.

For instance, it must hold for the location where a change in loads occurs (for $x = a$)

– the deflection calculated for the left-hand interval must equal the deflection calculated for the right-hand interval (continuity of the function) $\Rightarrow \quad w_I = w_{II}$

– the slope calculated for the left-hand interval must equal the slope calculated for the right-hand interval (continuity of the first derivative of the function, i.e. smoothness of the deflection curve) $\Rightarrow \quad \varphi_I = \varphi_{II}$

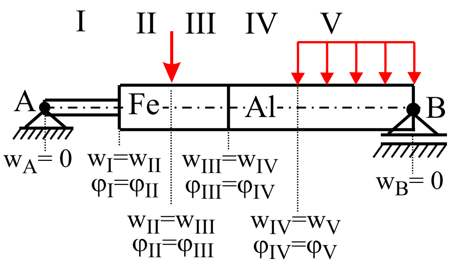


bar assumptions

At beams with the term M_{oy}/EJ_y expressed by different dependencies in various parts of the centreline, we can proceed as follows:

Problem 616

- We divide the centreline into intervals in which the term M_{oy}/EJ_y is expressed by a single (continuous and smooth) function dependency. The boundaries of intervals are in the locations with changes in loads, material or cross section characteristics.
- We formulate a differential equation for each of the intervals.
- We formulate the support deformation conditions resulting from the supports of the beam.



material characteristics

cross section characteristics

deflection curve

- For all the boundaries between intervals, we formulate the following two boundary conditions for the deflection curve:
  conditions of continuity (equality of deflections calculated for the left-hand and right-hand side intervals) $(w_i(a) = w_{i+1}(a))$,
  conditions of smoothness (equality of slopes calculated for the left-hand and right-hand side intervals) $(\varphi_i(a) = \varphi_{i+1}(a))$

Problem 622

Since two integration constants must be calculated for the solution to one differential equation, a corresponding number (two times the number of intervals) of boundary conditions are needed. For the correctness of these boundary conditions it is necessary to express the function $\dfrac{M_o(x)}{EJ_y}$ for all the intervals in the **same coordinate system**.

13.10.3. Integral approach

The deformation parameters in some particular points of the centreline can also be evaluated using Castigliano's theorem. věty. Castigliano's theorem

The strain energy accumulated in the beam with length l equals the sum of contributions of bending and shear: W_{M_o} W_T

$$W = W_{M_{oy}} + W_T = \frac{1}{2E}\int_0^l \frac{M_{oy}^2(x)}{J_y(x)}\mathrm{d}x + \frac{\beta}{2G}\int_0^l \frac{T^2(x)}{S(x)}\mathrm{d}x,$$

To solve the displacement of the point J of action of the force $\vec{F}_J$, we substitute the strain energy into Castigliano's theorem and differentiate the term in the general form: Castigliano's theorem

Example 625

$$w_J = \frac{\partial W}{\partial F_J} = \int_0^l \frac{M_{oy}}{EJ_y}\frac{\partial M_{oy}}{\partial F_J}\mathrm{d}x + \beta\int_0^l \frac{T}{GS}\frac{\partial T}{\partial F_J}\mathrm{d}x.$$

In this procedure, we must take into account that the deflection w_J is a global quantity (it depends on the deformations of all the beam or even of all the structure). Therefore the components of inner resultants must be expressed as function dependencies along all the length of the beam centreline. At long and slender beams ($l > 10h$) the contribution of the shear force can be neglected.

13.10.4. Comparison of differential and integral approaches

1) Differential approach:
It enables us

a) to solve also large deflections – using the equation valid for large deformations $\frac{\pm w''}{(1+w'^2)^{\frac{3}{2}}} = \frac{M_{oy}}{EJ_y}$ (only in certain simple cases), large deflections
b) to solve the magnitude of deflection and slope in any general points of the beam,
c) to evaluate the magnitude of the extreme deflection even in the case that we do not know the location of the extreme. Problem 624

Disadvantages: it does not comprehend the influence of shear force on the deflections and slopes, and it is usually more time-consuming and mathematically difficult.

2) Integral approach (Castigliano's theorem):
It enables us

a) to solve any deformation characteristic in a certain point of the centreline; if there is no corresponding load in the point in question, we add a complementary force $\vec{F}_d = 0$ or a complementary couple $\vec{\mathcal{M}}_d = 0$ and we solve the task in the same way like in the case of any other known external loads, Problem 618 Problem 621 characteristics
b) to comprehend the influence of the shear force $\vec{T}$ on deflections and slopes, Example 625
c) to carry out the calculation more easily and faster than using differential approach,
d) to choose any (advantageous) coordinate system which can be different in each of the centreline intervals,
e) to solve also curved and angular beams.

Disadvantages:

a) it can be used only within the range of linear elasticity (small strains and deflections, Hookean material, linear supports), linear elasticity
b) it solves only one deformation parameter in a particular point, it is difficult to be used in searching for extremes.

13.10.5. Supported beam

In the surroundings of supports there is a region where the assumptions of simple flection are not satisfied, because the support is not able to restrict only deflections and rotations of the centreline points. This region cannot be solved using the formulas valid for simple flection. If this region is decisive from the viewpoint of limit states, it should be solved e.g. using finite element method. bar assumptions

Procedure of how to solve supported beams

1. We isolate the beam as a free body and introduce reactions in the locations of the removed supports.
2. We formulate the applicable equations of static equilibrium.
3. We evaluate the degree of redundancy $s = \mu - \nu$. The following cases can occur:
   a) $s = 0$ – the bearing of the beam is statically determinate - we continue with par. 7. Problem 602
   b) $s \geq 1$ – the bearing of the beam is statically indeterminate - we continue with par. 4.

4. We create a released structure and formulate the compatibility equations, i.e. the support deformation conditions for deflections or rotations of centreline points, the number of which must equal the degree of static indeterminacy. released structure
5. We express the compatibility equations by means of loads using Castigliano's theorem. If the supports restrict longitudinal displacements of the bar, a non-zero normal force is created in the bar and the onefold loading changes into a combined one (flection + tension or compression). The compatibility equations can be: onefold loading
   a) homogeneous – the restricted kinematic parameter equals zero, Problem 617
   b) non-homogeneous – the restricted kinematic parameter differs from zero because of the production inaccuracies (e.g. different height of supports, misalignment of supports etc.), Problem 608
   c) circumstantial – the beam can be either statically determinate or indeterminate in dependence on the magnitude of deflection or rotation (e.g. a too high assembly clearance can disable the function of the support). Example 613
6. We formulate a set of equations consisting of conditions of static equilibrium of the beam isolated as a free body and compatibility equations of the released structure.
7. We solve the set of equations for support reactions.
8. We solve the stress state and deformation parameters in the same way as for a free beam.

13.11. Examples and problems

Examples

Problem 601 Problem 625 Problem 627

Problems

Problem 602 Problem 603 Problem 604 Problem 608 Problem 610

Problem 618 Problem 622 Problem 624 Problem 616 Problem 617