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Analysis:

Objective: to practise both of the possible (integral and differential) ways of evaluation of deformation parameters in a certain point of the bar..

approach

Classification of the bar: straight, with statically determinate bearing (one fixed support), loaded by an external concentrated force $\vec{F}$.

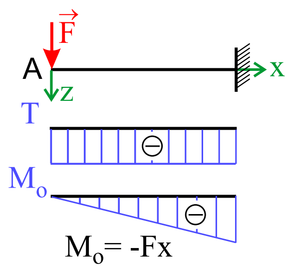
a) Integral approach:

coordinate system

basic flexion

principal central axis

The distribution of inner resultants shows us that there are two non-zero components of inner resultants in the bar, namely bending moment $\vec{M}_o$ (in y-direction) and shear force $\vec{T}$ (in z-direction). The bar is loaded by **basic flexion** (the line of action of the bending moment is identical with a principal central axis of the cross section), and the shear force creates, moreover, loading by shear; thus the bar can be called beam.



It can be proved that the influence of the shear force on the deflections of the beam is negligible if it holds $l > 10h$; therefore the strain energy will be calculated from the bending moment only.

The displacement of the point of action A of the force $\vec{F}$

proof

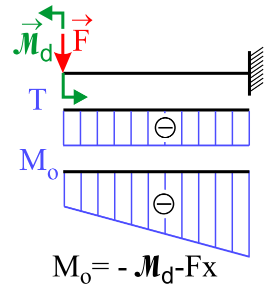
$$w = \frac{\partial W_{M_o}}{\partial F} = \frac{1}{EJ_y}\int_0^l M_o \cdot \frac{\partial M_o}{\partial F}\mathrm{d}x = \frac{1}{EJ_y}\int_0^l (-Fx)(-x)\mathrm{d}x = \frac{Fl^3}{3EJ_y},$$

where J_y is the second moment of the cross section related to the y-axis: $J_y = \dfrac{\pi d^4}{64}$.

displacement

J_y

The Castigliano's theorem enables us to calculate the slope of the bar in the point of action of a concentrated couple. However, there is no couple acting in the point A; we need to introduce an additional couple $\vec{\mathcal{M}}_d = 0$ here.



Castigliano's theorem

$$\varphi_A = \frac{\partial W}{\partial \mathcal{M}_d} = \frac{1}{EJ_y}\int_0^l M_o \frac{\partial M_o}{\partial \mathcal{M}_d}\mathrm{d}x = \frac{1}{EJ_y}\int_0^l (-\mathcal{M}_d - Fx)(-1)\mathrm{d}x$$

$$\varphi_A = \frac{1}{EJ_y}\int_0^l Fx\mathrm{d}x = \frac{Fl^2}{2EJ_y}$$

The resulting slope φ_A in the point A is positive, that means oriented according to the orientation of the additional couple $\vec{\mathcal{M}}_d$.

b) Differential approach:

Since the bending moment is described by a single equation $M_o = -Fx$ for all the beam centreline, we formulate only one differential equation of the deflection curve of the beam

deflection curve

$$w'' = -\frac{M_o}{EJ_y} = \frac{Fx}{EJ_y}$$

and we can obtain the result by its successive double integration

$$w' = \frac{Fx^2}{2EJ_y} + C, \qquad w = \frac{Fx^3}{6EJ_y} + Cx + D.$$

The obtained solution contains two integration constants that can be calculated using two boundary conditions, determined by the supports of the beam - the deflection and the slope in the fixed support equal zero.:

boundary conditions

$$w(l) = 0: \qquad 0 = \frac{Fl^3}{6EJ_y} + Cl + D$$

$$w'(l) = 0: \qquad 0 = \frac{Fl^2}{2EJ_y} + C$$

The calculated integration constants

$$C = -\frac{Fl^2}{2EJ_y}, \quad D = \frac{Fl^3}{3EJ_y}$$

can be substituted into the equation of the deflection curve

$$w = \frac{Fx^3}{6EJ_y} - \frac{Fl^2}{2EJ_y}x + \frac{Fl^3}{3EJ_y}$$

and the deflection and slope of the free end of the beam can be calculated by substitution $x = 0$:

$$w(0) = \frac{Fl^3}{3EJ_y}, \qquad w'(0) = -\frac{Fl^2}{2EJ_y}$$

We obtained the same result as by solution using Castigliano's theorem. The opposite signs are given by different sign conventions in both methods - Castigliano's theorem gives positive results in the orientation of the couple in question ($\vec{\mathcal{M}}_d$), while the slope calculated from the differential equation is positive if oriented clockwise.