16. Mathematical description of stress state

The stress state in a point of a body was defined as a set of general stresses in all sections containing this point. For an unambiguous numerical determination of the stress state it is sufficient to know the components of general stresses in three mutually perpendicular sections, which can be suitably ordered in a square matrix describing a stress tensor T_σ: stress state

$$T_\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

With respect to the symmetry of this tensor which is based on the assumption of small strains and shear stress equality theorem $(\tau_{ij} = \tau_{ji})$, only six of the stress tensor components are independent, i.e. three normal $(\sigma_x, \sigma_y, \sigma_z)$ and three shear $(\tau_{xy}, \tau_{xz}, \tau_{yz})$ stresses. equality τ

16.1. Principal coordinate system

First we mention an important property of all tensors, namely existence of a **principal coordinate system** in which the out-of-diagonal components of the tensor equal zero. The coordinate planes of the principal coordinate system are called **principal planes**. Thus no shear stresses $(\tau_{ij} = 0)$ but only normal stresses act in the principal planes of a stress tensor. We call them **principal stresses** and denote them by numerical subscripts according to the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$. principal c.s.

The stress tensor T_σ in the principal coordinate system has the following form:

$$T_\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Principal stress is normal stress in such a plane in which the shear stresses equal zero (i.e. general stress in the section is perpendicular to this section ($\vec{f}_\rho = \vec{\sigma}_\rho$)).

general stress

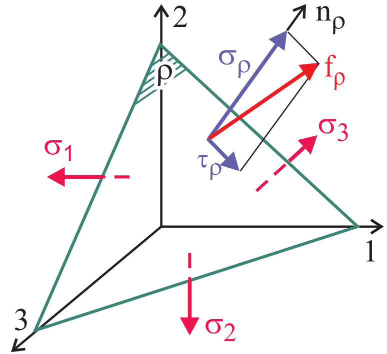
Principal stresses $\sigma_i (i = (1,\ 2,\ 3))$ can be calculated from the known components of the stress tensor T_σ determined in any general coordinate system; we calculate them by solving the **characteristic equation** of the stress tensor [1]:

$$\sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0,$$

where I_1, I_2, I_3 are invariants of the stress tensor determined by the following formulas:

$$I_1 = \sigma_x + \sigma_y + \sigma_z, \quad I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2, \quad I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

16.2. Calculation of stresses in a general plane



If we need to calculate stresses $\vec{f}_\rho, \sigma_\rho$ and τ_ρ in a general plane from the known principal stresses, it is advantageous to isolate an elementary tetrahedron with three faces in principal planes as a free body. There are three principal stresses σ_1, σ_2, σ_3 acting in the principal planes. The section ρ is defined by the base vector of the normal line $\vec{e}_\rho$, the components of which in the principal coordinate system are denoted as $\alpha_1, \alpha_2, \alpha_3$ (α_i – direction cosines of the normal of the plane ρ). From the static equilibrium equations of the element we obtain - after neglecting volumetric forces - the following relations for components of general stress in the section ρ:

$$f_{\rho_1} = \sigma_1 \alpha_1, \quad f_{\rho_2} = \sigma_2 \alpha_2, \quad f_{\rho_3} = \sigma_3 \alpha_3$$

We can simplify them by transcription in a matrix form:

$$f_\rho = T_\sigma \cdot \alpha, \qquad \begin{pmatrix} f_{\rho_1} \\ f_{\rho_2} \\ f_{\rho_3} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Magnitude of the general stress can be calculated using the following relation for magnitude of a vector:

$$f_\rho = \sqrt{f_{\rho_1}^2 + f_{\rho_2}^2 + f_{\rho_3}^2} = \sqrt{\sigma_1^2 \alpha_1^2 + \sigma_2^2 \alpha_2^2 + \sigma_3^2 \alpha_3^2}$$

To predict the risk of limit states (failure), we often need to know the normal ($\vec{\sigma}_\rho$) and shear ($\vec{\tau}_\rho$) components of the general stress $\vec{f}_\rho$.

Magnitude of the normal stress can be calculated as a projection of the general stress f_ρ into the normal direction of the plane ρ:

$$\sigma_\rho = \vec{f_\rho} \cdot \vec{e}_\rho = \sigma_1 \alpha_1^2 + \sigma_2 \alpha_2^2 + \sigma_3 \alpha_3^2$$

Determination of the shear stress would be more complex in this way because we do not know the direction in the plane ρ in which the shear stress acts. The direction of the shear stress, however, is not significant in analysis of limit states at isotropic materials. Therefore we can calculate only the magnitude of the shear stress using Pythagoras' formula (see figure at the previous page)

$$\tau_\rho = \sqrt{f_\rho^2 - \sigma_\rho^2},$$

into which we substitute the calculated magnitudes of stresses f_ρ and σ_ρ.

(see figure)

16.3. Stresses in octahedric plane

Among the various sections ρ containing the investigated (dangerous) point, the **octahedric plane** is one of the most important from the viewpoint of limit states prediction; the normal of this plane forms the same angles α'_o with all the principal axes 1, 2, 3, so that all the three direction cosines α_o are also equal:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_o, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 3\alpha_o^2 = 1, \quad \Rightarrow \quad \alpha_o = \frac{1}{\sqrt{3}}$$

The von Mises (HMH) plasticity condition is just based on shear stress in this plane. HMH
The magnitudes of the general, normal and shear stresses in the octahedric plane can be calculated by substituting the direction cosines of the octahedric plane into the relations for stresses in a general section ρ: stress in ρ

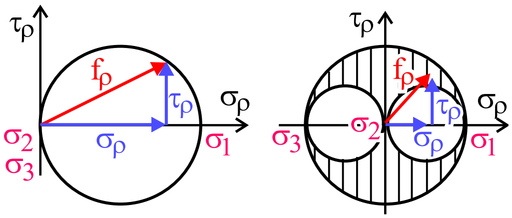
$$f_o = \sqrt{\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)} \qquad \sigma_o = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\tau_o = \sqrt{f_o^2 - \sigma_o^2} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

16.4. Graphical representation of stress state

In this chapter we introduce another important property of tensors which is the possibility of their graphical representation; it is carried out in the Mohr's plane where diagonal components of the tensor (i.e. components on the principal diagonal of the matrix representing the tensor) are represented as abscissa (horizontal coordinate – normal stresses in the case of a stress tensor) and out-of-diagonal components are represented as ordinate (vertical coordinate - shear stresses in the case of a stress tensor).

Uniaxial stress state was represented in this way in the chapter concerning simple tension and shear stress state was represented in the chapter concerning simple torsion. As it is evident from the figure, the radius vector of a point in the Mohr's plane of stresses determines the general stress f_ρ in the given section ρ, defined by the components σ_ρ and τ_ρ.



tension

torsion

The following statement was proven in literature (e.g. in [1]).

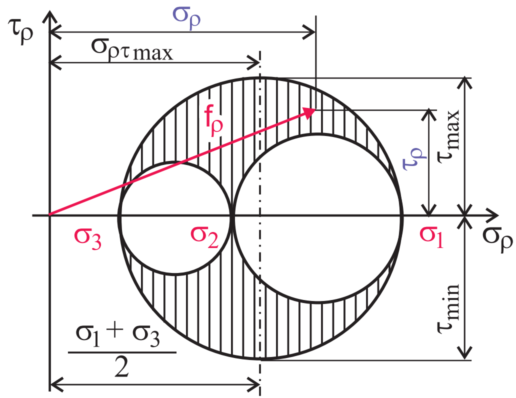
If the stress state in a point of the body is determined by principal stresses σ_1, σ_2, σ_3, the points representing the general stresses f_ρ (ρ is any plane containing the investigated point) lie in the hatched region of the Mohr's plane among the three Mohr's circles including the boundary.

The hatched region in the figure including also all the three boundary circles represents thus the stress state in the point of the body.

Principal stresses σ_1 and σ_3 are the extreme normal stresses in the point of the body, extreme shear stresses are

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = -\tau_{min}$$

and they act in those sections where normal stress equals $\sigma_{\rho\tau_{\max}} = \frac{\sigma_1 + \sigma_3}{2}$



stress state

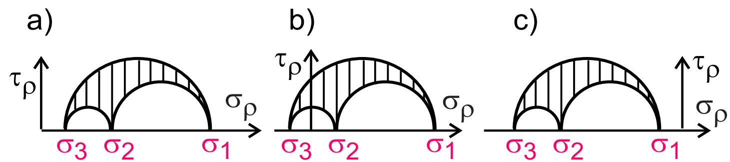
16.5. Specific types of stress states

In the previous chapters, we analyzed stress states in which the principal stresses were mutually different and all of them had a non-zero magnitude, i.e. $\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq 0$. However, we often meet cases in which some of the principal stresses are zero or mutually equal. The Mohr's representation offers us a fast and illuminating idea on the stress state, incl. the extreme components of stresses.

16.5.1. Triaxial stress state

1) **general**

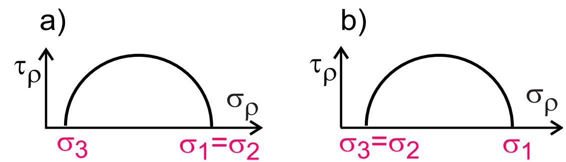
$$\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq 0$$



2) **half-uniform**

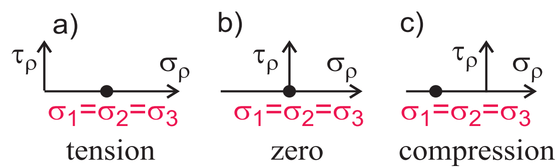
a) $\sigma_1 = \sigma_2 \neq 0, \quad \sigma_3 \neq 0$

b) $\sigma_2 = \sigma_3 \neq 0, \quad \sigma_1 \neq 0$



3) **uniform**

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma$$



In the case of the uniform triaxial stress state, there is no shear stress in any section. Therefore limit state of elasticity cannot occur under the condition of this stress state; it is evident from the plasticity criteria which are based on some shear stress component in all cases.

plasticity criterion

16.5.2. Biaxial (plane) stress state

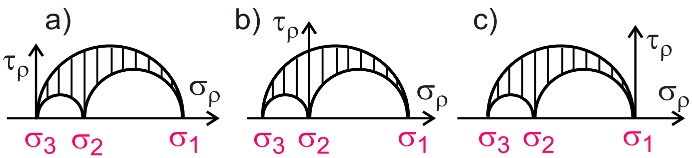
one of the principal stresses equals zero

1) **general**

a) $\sigma_3 = 0, \quad \sigma_1 \neq \sigma_2 \neq 0$

b) $\sigma_2 = 0, \quad \sigma_1 \neq \sigma_3 \neq 0$

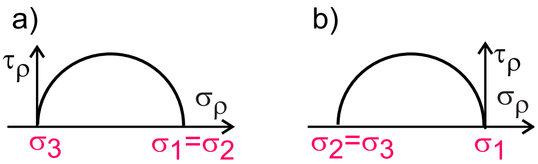
c) $\sigma_1 = 0, \quad \sigma_2 \neq \sigma_3 \neq 0$



2) **uniform** (equibiaxial stress)

a) $\sigma_3 = 0, \quad \sigma_1 = \sigma_2 \neq 0$

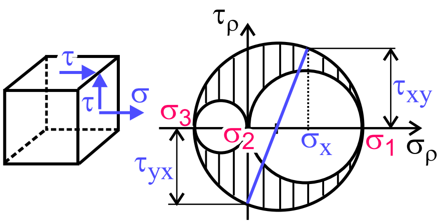
b) $\sigma_1 = 0, \quad \sigma_2 = \sigma_3 \neq 0$



3) **bar-type**

strain assumptions

This type of stress state can be found in all bars (i.e. rod-like bodies, e.g. columns, beams, shafts etc.), therefore we analyze it now in greater detail. This stress state is determined by normal and shear stress components in the cross section of the bar



$$\sigma_x = \sigma \neq 0, \tau_{xy} = \tau \neq 0,$$

while all the other components of the stress tensor equal zero. Let's substitute this stress components into the characteristic equation of the stress tensor to calculate the principal stresses which will be necessary for prediction of limit states (failures):

$$\sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma \qquad I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 = -\tau^2$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = \begin{vmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \;\Rightarrow\; \sigma(\sigma^2 - I_1\sigma + I_2) = 0 \;\Rightarrow\; \sigma_I = 0,\; \sigma_{II,III} = \frac{I_1}{2} \pm \sqrt{\left(\frac{I_1}{2}\right)^2 - I_2}$$

after substituting $I_1 = \sigma$ and $I_2 = -\tau^2$ we obtain

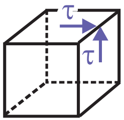
$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \quad \sigma_2 = 0, \quad \sigma_3 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}.$$

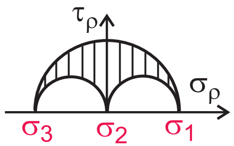
As the square root expressing the radius of the Mohr's circle is always positive, it reads $\sigma_1 \geq 0$ and $\sigma_3 \leq 0$, so that the calculated stresses meet the relation $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

4) **pure shear**

It is a special case of the plane or bar-type stress states for all the normal stresses are zero $\sigma_i = \sigma = 0$. Then it reads for principal stresses

$$\sigma_1 = -\sigma_3 = \tau, \qquad \sigma_2 = 0.$$





torsion

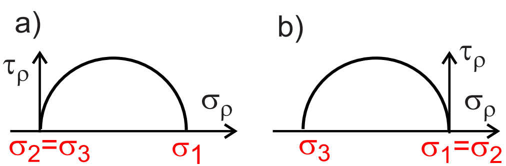
This type of stress state occurs e.g. in bars under simple torsion.

16.5.3. Uniaxial stress state

Two of the principal stresses equal zero

a) tensile $\sigma_1 > 0, \quad \sigma_2 = \sigma_3 = 0$

b) compressive $\sigma_3 < 0, \quad \sigma_1 = \sigma_2 = 0$



16.5.4. Zero stress (stress-free) state

$$\sigma_1 = \sigma_2 = \sigma_3 = 0$$