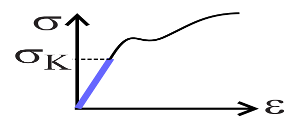
7. Basic formulations of linear theory of elasticity

The division of the theory of elasticity to **linear** and **non-linear** elasticity is given by the shape of the dependance of deformation and stress parameters on the loads.

Linear theory of elasticity searches into stress and deformation states under the assumption that all the dependencies among loads, stress states, strain states and displacements are linear. If linearity is violated in any of these dependencies, the resulting problem is non-linear and it must be solved using methods of the **non-linear theory of elasticity**.

The distinction between linear and non-linear elasticity is fundamental for stress-strain analyses and for judging of structures as well. The **solutions to linear problems** are much easier but their **practical applicability** is limited.

The necessary conditions for a problem to be linear:



- material is linear elastic,
- displacements of all points are small in comparison with the dimensions of the investigated bodies,
- components of strain tensor are small ($\ll 1$, usually on the order of 10^{-3} or less),

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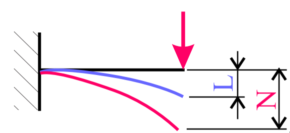
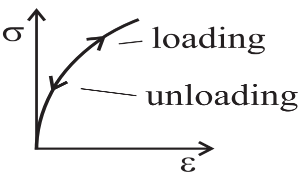


- boundary conditions are linear.

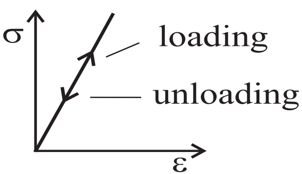
In this course we will deal with **problems** in which deviations from linear behaviour are not substantial.

7.1. Hooke's law

Let's note that „elastic deformation" means fully reversible deformation. As a consequence of deformation reversibility, the deformation in a certain time depends only on the loads acting in this time and does not depend on the history of loads. Consequently stress state of the body is also determined only by loads acting in this time.



The dependence between strain ε and stress σ is generally non-linear. This non-linearity makes the stress-strain analysis much more difficult.



At steel as the most common technical material used in design of structures, however, this dependence can be approximated by a **linear** dependence in the whole range of elastic deformations with a sufficient accuracy. In this way we obtain the simplest *computational model of elastic material* behaviour – **linear elastic (Hookean) material**, which obeys Hooke's law.

Hooke's law represents the simplest form of constitutive (physical) equations. These equations describe the relations between the components of strain tensor T_σ and stress tensor T_ε in the investigated point of the body. If the material is linear elastic, then all these relations are linear. If the state of stress in a point od the body is uniaxial (e.g. in the central part of the specimen during tension or compression tests), the stress tensor T_σ has only one non-zero component - the normal stress σ_x in the longitudinal specimen direction (x axis) and the dependence between strain and stress components in this direction is described by the following linear relation

$$\sigma_x = E\varepsilon_x,$$

where E is a proportionality parameter called **Young's modulus** or **modulus of elasticity in tension** (the magnitude of this modulus in compression is the same at most materials). As in tension or compression tests also lateral dimensions of the specimen change, (the **state of strain** is not uniaxial but triaxial), not only longitudinal but all the length strains are non-zero and they can be calculated using the formula:

$$\varepsilon_y = \varepsilon_z = -\mu\varepsilon_x,$$

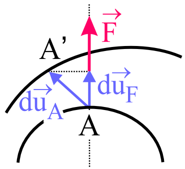
where the parameter μ is called **Poisson's ratio**. As at isotropic materials (i.e. materials with the same properties in all directions) no shear strains occur in tension or compression tests, the above relations define all the components of the strain tensor. Thus only two elastic parameters (both of them can be determined from a single test, in tension, or in compression) are sufficient to the complete description of linear elastic isotropic material behaviour. At an anisotropic material, the material properties depend on the direction and 21 independent elastic parameters are necessary for a complete description of linear elastic properties of the most general anisotropic material. However, the above relations are not sufficient for description of a linear elastic isotropic material properties, because their validity is limited to the case of uniaxial stress state. For a general triaxial stress state, all the normal stresses are functions of all the length strains and conversely. These relations are described by a general Hooke's law, from which another form of Hooke's law, valid for the shear stress state (in plane) only, can be derived:

$$\tau = G\gamma.$$

The proportionality parameter G in this relation is called **modulus of elasticity in shear** or **shear modulus**. Normally it does not need to be measured at isotropic materials, because it can be calculated from the above two elastic constants using another relation which can be derived from the general Hooke's law:

$$G = \frac{E}{2(1+\mu)}\,.$$

7.2. Work done by a force during body deformation



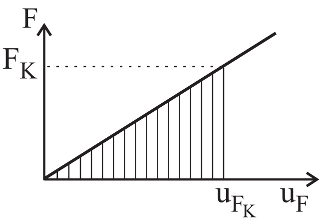
Every force, the point of action of which moves, does some work. In general, this work can be expressed by the following formula:

$$A_F = \int_u \vec{F} \mathrm{d}\vec{u}_A = \int_{u_F} F \mathrm{d}u_F,$$

where the vector $\mathrm{d}\vec{u}_A$ represents an elementary displacement of the point of action of force $\vec{F}$ and $\mathrm{d}u_F$ is the projection of this displacement in the direction of force F. The integral (it means the work done by the force) can be calculated only under condition that the dependence of the force on the position is known.

Let's assume that the only one isolated force is acting on the linear elastic body in point A. The body is deformed in consequence of the loading external force, so that the external force is in equilibrium with internal forces induced in the body; therefore this force must also change linearly with the change of position $F(u_F) = c \cdot u_F$ in the whole range of instantaneous values $u_F \in \langle 0; u_{F_K} \rangle$, so that it increases from the zero initial value to the final value $F_K = c \cdot u_{F_K}$. During this process, this changing force does the work:

$$A_F = \int_0^{u_{F_K}} F \mathrm{d}u_F = \int_0^{u_{F_K}} c u_F \mathrm{d}u_F = \frac{c u_{F_K}^2}{2} = \frac{F_K^2}{2c} = \frac{1}{2} F_K u_{F_K}.$$



The geometrical interpretation of the integral corresponds to the area below the curve in the graph $F = F(u_F)$; in the case of linear dependence between force and displacement, the work done by the force corresponds to the triangle hatched in the graph.

If there are also other forces acting on the investigated body, the position of the point of action of force $\vec{F}$ can be changed also in consequence of their influence. We can also calculate the work done by force $\vec{F}$ in consequence of changes of the other forces (the force $\vec{F}$, however, remains constant during this process). This work done by the constant force $\vec{F}$ on the displacement u_F of its point of action (in the direction identical with that of the force) from the zero point to u_{F_K} is

$$A_F = \int_0^{u_{F_K}} F_K \mathrm{d}u_F = F_K u_{F_K}.$$

The graphic interpretation of this integral is a rectangular area and we really obtained the corresponding result.

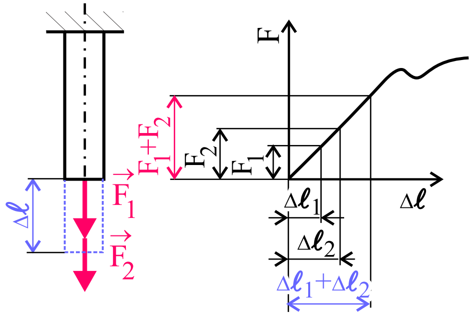
7.3. General principles of the linear theory of elasticity

There are several fundamental principles valid in linear theory of elasticity. Now, we present some of them.

7.3.1. Principle of superposition

Example:

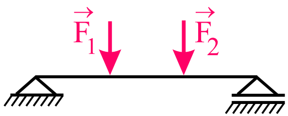
A bar is loaded by two isolated forces $\vec{F}_1$ and $\vec{F}_2$. The elongation of the bar equals the sum of elongations caused by the individual forces ($\Delta l = \Delta l_1 + \Delta l_2$). Warning: the principle is valid only in the linear part of the stress-strain diagram (linear theory of elasticity); e.g. principle of superposition is not valid for grey cast iron, because its tension diagram is non-linear from the very beginning. The principle can be extended for any loads (not only forces but couples and distributed loads as well).



Stress (deformation) state of the body loaded by a system of **loads** equals - in linear theory of elasticity - the sum of stress (deformation) states caused by the individual **loads** of the system.

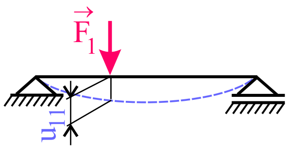
7.3.2. Principle of reciprocity of works (Betti's theorem)

Let's have a beam loaded by a system of loads given by a set of two isolated forces $\vec{F}_1$ and $\vec{F}_2$. The beam deforms under load and the points of action of the both forces shift.



Let's denote the displacement of the force $\vec{F}_i$ along its line of action caused by the action of the force $\vec{F}_j$ by symbol u_{ij}; the meanings of subscripts at work A_{ij} are analogical. Let's analyse two load histories:

1. First, the body is loaded by force $\vec{F}_1$ and then force $\vec{F}_2$ is added.

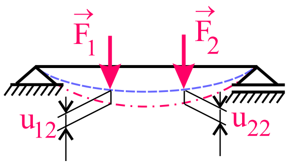


During the loading $\left\{\vec{0}\right\} \to \left\{\vec{F}_1\right\}$, force $\vec{F}_1$ does work A_{11} given by the formula

$$A_{11} = \frac{1}{2}F_1 u_{11}.$$

Analogically, during the loading $\left\{\vec{F}_1\right\} \to \left\{\vec{F}_1\right\} \cup \left\{\vec{F}_2\right\}$, force $\vec{F}_2$ does work

$$A_{22} = \frac{1}{2}F_2 u_{22},$$



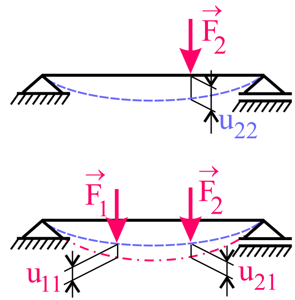
and at the same time, as the force $\vec{F}_2$ induces displacements of all the points of the body (except of the immovable ones because of supports), the force $\vec{F}_1$ does the work

$A_{12} = \int\limits_{u_{11}}^{u_{11}+u_{12}} F_1 \mathrm{d}u_{12} = F_1 u_{12}$ and the total work is

$$A_1 = A_{11} + A_{22} + A_{12} = \frac{1}{2}F_1 u_{11} + \frac{1}{2}F_2 u_{22} + F_1 u_{12}.$$

2. Now let's consider an opposite procedure. First the body is loaded by force $\vec{F}_2$ and then force $\vec{F}_1$ is added.

The work can be obtained in the similar way:



$$A_2 = A_{22} + A_{11} + A_{21} = \frac{1}{2}F_2u_{22} + \frac{1}{2}F_1u_{11} + F_2u_{21}.$$

As neither stress state nor deformation under load depend on the history of loads (if the body is linear elastic), nor the deformation work depends on the history (it means that the system of loads is conservative, i.e. preserving work). Therefore the following equality must be valid:

$$A_1 = A_2.$$

After substitution we obtain:

$$\frac{1}{2}F_1u_{11} + \frac{1}{2}F_2u_{22} + F_1u_{12} = \frac{1}{2}F_2u_{22} + \frac{1}{2}F_1u_{11} + F_2u_{21}$$

and after simplification

$$\boxed{F_1u_{12} = F_2u_{21}}\ .$$

This equality expresses the simplest form of the Betti's theorem. In words, it can be expressed as follows:

Betti's theorem:
If two forces $\vec{F}_1$ and $\vec{F}_2$ are acting on a linear elastic body, the work done by the force $\vec{F}_1$ on the displacement components induced by the force $\vec{F}_2$ equals to the work done by the force $\vec{F}_2$ on the displacement components induced by the force $\vec{F}_1$.

Naturally, the theorem can be generalised for more complex systems of loads. However, its simplification by introduction of unit forces is more important for us. If both forces equal 1 N, they can be cancelled out in the equation. The corresponding coefficients are called **causal coefficients** and the following formula is valid for them:

$$\eta_{12} = \eta_{21}.$$

In accordance with the above notation of the displacements, the causal coefficient η_{12} means displacement of the point of action of force $\vec{F_1}$ caused by the unit force $\vec{F_2}$. These causal coefficients are already characteristic constants for the body and its given points. They can be advantageously exploited in calculation of the displacement of the point of action of the force, when the body is loaded by a system of loads. For example, the displacement of the force $\vec{F_1}$ when the body is loaded by a system of two forces $\left\{\vec{F_1}; \vec{F_2}\right\}$ can be calculated using the formula

$$u_1 = F_1\eta_{11} + F_2\eta_{12}.$$

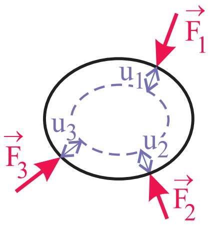
7.3.3. Deformation work of a system of isolated forces

A system of isolated forces $\Pi = \{\vec{F}_1, \vec{F}_2\}$. acts on a linear elastic body. As the deformation work is independent of the load history, the sequence of the loads can be chosen in the way that the force $\vec{F}_1$ acts on the body at first, then the force $\vec{F}_2$ is added etc. Then the deformation work equals:

$$\begin{aligned}
\{\vec{0}\} \to \{\vec{F}_1\} \quad &\Rightarrow \quad A_1 = \tfrac{1}{2}F_1 u_{11}.\\
\{\vec{0}\} \to \{\vec{F}_1\} \to \{\vec{F}_1\} \cup \{\vec{F}_2\} \quad &\Rightarrow \quad A_2 = A_1 + \tfrac{1}{2}F_2 u_{22} + F_1 u_{12} =\\
&\qquad\;\; = \tfrac{1}{2}F_1(u_{11}+u_{12}) + \tfrac{1}{2}F_2 u_{22} + \tfrac{1}{2}F_1 u_{12}.
\end{aligned}$$

Using Betti's theorem we can obtain

$$F_1 u_{12} = F_2 u_{21} \quad \Rightarrow \quad A_2 = \frac{1}{2}F_1(u_{11}+u_{12}) + \frac{1}{2}F_2(u_{21}+u_{22})$$



As displacement u_i can be expressed as $u_i = u_{i1} + u_{i2}$, we obtain for the work of the whole system

$$A = \frac{1}{2}F_1 \sum_{i=1}^{2} u_{1i} + \frac{1}{2}F_2 \sum_{i=1}^{2} u_{2i} + \cdots = \frac{1}{2}\sum_{i=1}^{2} F_i u_i,$$

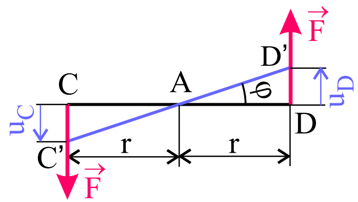
where u_i is the total displacement of the point of action of the force $\vec{F}_i$, in the direction of its line of action as the consequence of both forces acting on the body. The sum can be naturally generalised for any number of forces.

If a set of forces $\Pi = \{\vec{F}_1, \vec{F}_2, \cdots \vec{F}_n\}$ acts on a a linear elastic body, and the displacements of their points of action $A_1, A_2, \cdots A_n$ in their directions are denoted as $u_1, u_2, \cdots u_n$, then the total work can be calculated using the following formula:

$$A = \frac{1}{2}F_1u_1 + \frac{1}{2}F_2u_2 + \cdots + \frac{1}{2}F_nu_n = \frac{1}{2}\sum_{i=1}^{n} F_iu_i.$$

Deformation work of couple of forces

A couple of forces defined by the momentum $\vec{\mathcal{M}}$ (with magnitude $\mathcal{M} = 2rF$) acts on a linear elastic body. The displacements of the points of action of the forces of the couple can be expressed in the form $u = r \ \mathrm{tg}\ \varphi$ and for a small angle (which is an assumption of linear theory of elasticity) it reads $u = r\varphi$. The work done by the couple of forces is



$$A = \frac{1}{2}Fu_\mathrm{C} + \frac{1}{2}Fu_\mathrm{D} = \frac{1}{2}Fr\varphi - \frac{1}{2}F(-r\varphi) = \frac{1}{2}F2r\varphi = \frac{1}{2}\mathcal{M}\varphi$$

a) Turning angle φ in point A of the body is determined by the changes in direction angles of a straight line fixed to the body in point A.

b) Deformation work done by the couple of isolated forces is: A= $\frac{1}{2}\mathcal{M}\varphi$ where φ is the turning angle in the plane of the couple of forces between the deformed and undeformed states.

7.3.4. Castigliano's theorem

We present a simplified derivation of Castigliano's theorem for a beam. Let's have a beam loaded by two forces in accordance with chapter 7.3.2. The deformation work A done during its loading process (for a beam made of an elastic material, this work equals the reversible strain energy W) is linear function of the acting forces, which was derived above in the form

$$A = W = \frac{1}{2}F_1u_1 + \frac{1}{2}F_2u_2.$$

Both displacements u_1 and u_2 of points of action of the loading forces are also linear functions of both of the forces. These displacements can be expressed using causal coefficients η in the form

$$u_1 = F_1\eta_{11} + F_2\eta_{12} \qquad\qquad u_2 = F_2\eta_{22} + F_1\eta_{21}$$

The meaning of causal coefficients was explained in the chapter 7.3.2. Betti's theorem. By substitution into the above equation for deformation work we obtain the following formula for strain energy

$$W = \frac{1}{2}\left(F_1^2\eta_{11} + F_1F_2\eta_{12} + F_2^2\eta_{22} + F_1F_2\eta_{21}\right),$$

which can be easily differentiated with respect to any of the forces (causal coefficients η_{ij} are constants for a given body and given points). For example, by differentiation with respect to the force F_1 we obtain:

$$\frac{\partial W}{\partial F_1} = \frac{1}{2}\left(2F_1\eta_{11} + F_2\eta_{12} + F_2\eta_{21}\right).$$

This manipulation is based on the mutual independence of forces (that means $\frac{\partial F_1}{\partial F_2} = 0 = \frac{\partial F_2}{\partial F_1}$.) As the causal coefficients are independent from the sequence of subscripts ($\eta_{12} = \eta_{21}$ as a consequence of Betti's theorem), the above relation can be manipulated to obtain the form:

$$\frac{\partial W}{\partial F_1} = \frac{1}{2}\left(2F_1\eta_{11} + 2F_2\eta_{12}\right) = u_1.$$

If we generalise this formula for the j-th force of a system of loads, we obtain the first part of the Castigliano's theorem:

$$u_j = \frac{\partial W}{\partial F_j}.$$

If the beam is loaded moreover by a couple of forces $\mathcal{M}_j$ then, during the loading process, this couple does the work

$$A = W = \frac{1}{2}\mathcal{M}_j\varphi_j,$$

where φ_j is the turning angle of the straight line fixed to the body in the point of action of the momentum $\mathcal{M}_j$. Then under condition of the mutual independence of the moments and forces we can obtain the second part of Castigliano's theorem by the same manipulations in the form:

$$\varphi_j = \frac{\partial W}{\partial \mathcal{M}_j}.$$

Verbally the Castigliano's theorem can be expressed as follows:

For a linear structure the partial derivative of the total strain energy with respect to any force $\vec{F}_J$ is equal to the displacement u_J of the point of action of this force in the direction of its action (provided that the strain energy is expressed as a function of the forces): $u_J = \frac{\partial W}{\partial F_J}$.
The partial derivative of the total strain energy with respect to any couple $\vec{\mathcal{M}}_J$ is equal to the corresponding turning angle φ_J of the point of action of this couple in the direction of its action (provided that the strain energy is expressed as a function of the forces and couples): $\varphi_J = \frac{\partial W}{\partial \mathcal{M}_J}$.

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The Castigliano's theorem is the most important theorem of linear theory of elasticity from the viewpoint of practical application, because it enables us to calculate deformation characteristics of any linear elastic body, provided that we are able to formulate a relation for its strain energy. The whole system of bodies must be included in the strain energy if the deformations of the neighbouring bodies (or of the frame) are not negligible in comparison with the deformations of the investigated body.

Note:

A negative sign of the displacement (or turning angle) means that the orientation of this displacement (angle) is opposite to the orientation of the corresponding force (couple of forces). Castigliano's theorem is independent from sign conventions, because a positive work means always that the displacement is oriented according to the orientation of the acting force.