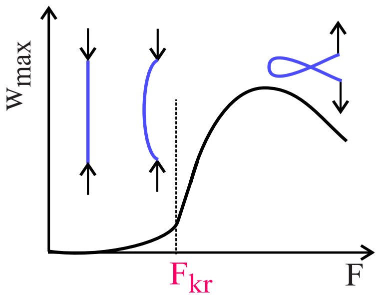
15. Buckling and stability

One of the simple loads we dealt with above was **simple compression**. One of the basic assumptions was that cross sections of the bar mutually **only come near** (or draw apart in the case of tension). compression

However, if a slender bar is loaded in compression (then we call it usually a column) its real behaviour is different. Beyond a certain load the column begins to **bend** and this **bending** is as pronounced that it becomes the substantial type of deformation. It means that the **type of substantial deformation changes** during the loading process. While in the initial phase of the loading, **shortening** of the column is substantial (and no or only a negligible bending occurs), under higher loads the situation is opposite - the **bending** is substantial and the shortening becomes less important. The boundary between these two phases is denoted as **limit state of shape stability or buckling**. limit state



The limit state of buckling is the state in which the type of the substantial deformation of the column is changed.

Problem 701

15.1. Buckling of an ideal free column

We analyse a column under the following assumptions:

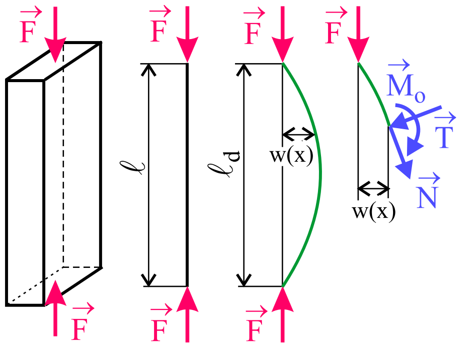
a) the centreline of the column is perfectly straight in the unloaded state,
b) the column is prismatic and non-screw-shaped,
c) the cross section is thick-walled (all the dimensions of the cross section are on the same order),
d) the column is loaded by two isolated forces F (being in static equilibrium) acting in centroids of the column facings; the lines of action of these forces are identical with the centreline of the column in the unloaded state,
e) the material of the column is homogeneous. isotropic and perfectly (without any limitations) linear elastic ($\sigma_K \rightarrow \infty$),
f) during the whole loading process the bar assumptions of simple loading are valid. bar assumptions

Meeting the above assumptions is characteristic for ideal loading of an ideal bar.
The objective of the solution is, in the first place, to determine when shortening is the substantial deformation of the column an when, in opposite, bending is more significant. Therefore we limit ourselves on the significant components of inner resultants:

shortening - normal force $\vec{N}$
bending - bending moment $\vec{M}_o$

Stress states and deformations of the bar under compression without bending have been analysed in chapter 11. Simple tension and compression. Now we focus on flection. compression
From the above methods of calculation of deformation under flection, only the differential equation of the deflection curve can be applied, namely in the form valid for large deformations.

Moreover, for the isolation of an element of the bar as a free body, the **deformed** shape of the bar must be taken into account (because in the undeformed shape, there is no bending moment!).
As a consequence of the deflection of the deformed centreline, shear force $\vec{T}$ and bending moment $\vec{M}_o$ act in the cross sections (in addition to the normal force $\vec{N}$). Therefore the bar is loaded by a combination of compression, shear and flection; as it is, however, long and slender (otherwise there would be no bending), bending load is the most substantial, while the other two components of inner resultants are negligible.



deflection curve

element

As we assumed the bar to be homogeneous, prismatic and non-screw-shaped, the deflection curve will be an in-plane curve.

We express the bending moment $M_o(x)$ from the moment equilibrium equation of the bar element and substitute it in the differential equation of the deflection curve valid for large deformations:

deflection curve

$$M_o(x) - Fw(x) = 0 \qquad \Rightarrow \qquad M_o(x) = Fw(x)$$

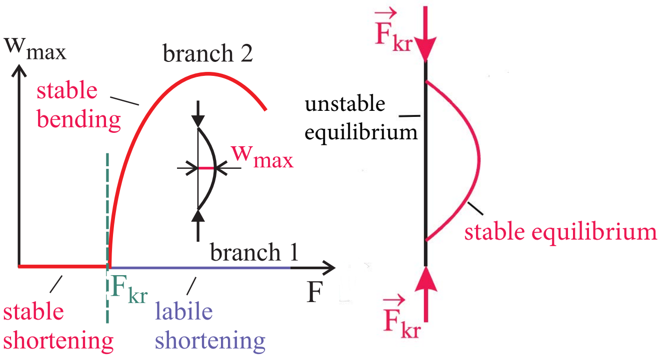
It is evident from the relation that the bending moment and, consequently, stresses in the bar are function of the deflection w; therefore stresses and deformations cannot be solved separately as it was possible in the linear theory of elasticity.

$$\frac{w''}{(1+w'^2)^{3/2}} = -\frac{Fw(x)}{EJ}$$

The general solution to the above differential equation of the 2nd order includes two integration constants; we must formulate two boundary conditions to determine their values.

$$\begin{aligned} x &= 0 \qquad & w &= 0 \\ x &= l_d \qquad & w &= 0 \end{aligned}$$

The solution to the differential equation with the above boundary conditions is not realisable, because we do not know the real distance l_d between the ends of the bar which is less than the length l of the bar. Lagrange solved this problem by neglecting this **difference** (for $l = l_d$). We introduce only the result of the Lagrangean solution in the form of dependence of the maximum deflection $w_{\max}$ on the load F.



In the figure you can see the critical force of buckling F_{kr} defining the following intervals:

F	$<$	F_{kr}	-	the bar is shortened only, there is no deflection,
F	$>$	F_{kr}	-	the bar is either only shortened, then it is in a labile equilibrium (branch No 1 in the figure), or only bended and then it is in a stable equilibrium (branch No 2 in the figure),
F	$=$	F_{kr}	-	the stable shortening changes into instable and bending becomes the stable deformation state; it is the point of equilibrium bifurcation.

The point of **equilibrium bifurcation** represents the limit state of buckling of an ideal bar under ideal compressive load.

The Lagrangean solution is mathematically very difficult and not suitable for practical use. Therefore we solve the above differential equation of the deflection curve under assumption of small deformations ($w' \ll 1 \Rightarrow 1 + w'^2 \doteq 1$). It is evident from the figure that this assumption can be valid until the force reaches the critical value F_{kr}; in this range deflections are negligible. It is not possible to determine the deflections of the bar after its buckling by this simplified solution but only the values of the critical force at which the buckling (limit state) occurs. Thus we solve the differential equation of the deflection curve in the form: limit state

$$w'' + \frac{Fw(x)}{EJ} = 0.$$

As it holds $l_d \doteq l$, the boundary conditions can be expressed in the form

$$\begin{array}{rclcrcl} x & = & 0 & \qquad & w & = & 0 \\ x & = & l & \qquad & w & = & 0 \end{array}$$

By denoting $p^2 = \frac{F}{EJ}$ it can be transformed into the normalised form

$$w'' + p^2 w = 0,$$

for which the solution is known, among others, in the goniometric form

$$w = C_1 \sin(px) + C_2 \cos(px).$$

Integration constants can be determined from the above boundary conditions:

$$\begin{array}{rclcrclcl} w(0) & = & 0: & \qquad & 0 & = & C_1 \sin 0 + C_2 \cos 0 & \Longrightarrow & C_2 = 0 \\ w(l) & = & 0: & \qquad & 0 & = & C_1 \sin(pl) & \Longrightarrow & C_1 \sin(pl) = 0 \end{array}$$

The second condition will be met if

a) $C_1 = 0 \implies w = 0$ with any sinus function argument $\implies$ the bar remains straight under any load F, it corresponds to the branch No 1 of the Lagrangean solution (labile equilibrium); this situation cannot occur in practice because of imperfections of a real bar, but we can obtain it as an unrealistic result of a numerical solution (e.g. using FEM).

b) $C_1 \neq 0 \implies \sin(pl) = 0 \implies pl = k\pi$ for $k = 0, 1, 2, \ldots$

We substitute for p : $\quad l\sqrt{\dfrac{F}{EJ}} = k\pi \implies F = \dfrac{(k\pi)^2 EJ}{l^2} \quad$ for $k = 0, 1, 2, \ldots$

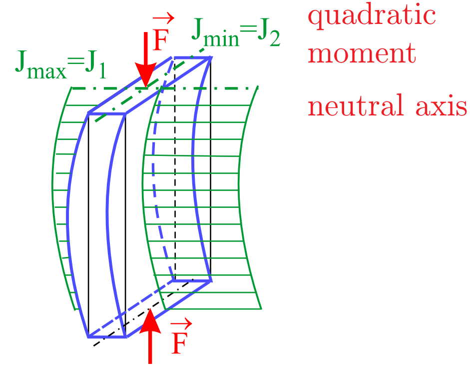
- $k = 0$: $\quad F = 0$ – the bar is unloaded in this case and there is no reason for it to deform, therefore $w = 0$.
- $k = 1$: $\quad F = F_{kr} = \dfrac{\pi^2 EJ}{l^2} \neq 0 \implies w \neq 0$, the deflection is uncertain, because the condition is met for any C_1 value. By comparison of this result with the general solution we can see that they are in agreement in the surroundings of the point F_{kr} for very low deflections, because the tangent line to the curve of w dependence is perpendicular to the F axis in this point. It means, however, that we obtained the exact critical force value F_{kr} from this approximate solution (valid however for an ideal and ideally loaded bar only).
- $k > 1 \Rightarrow F > F_{kr}$ and the deformation state (see figure) would be unstable so that it cannot occur in a real structure.

The stability of this deformation state can be achieved in a real structure by supports (deflection constraints) in some points of the bar; the critical force value can be substantially increased in this way. At a free bar, however, the only stable state is the deflected state under load F_{kr}.

The above analysis enables us to formulate the following conclusion:

If we solve the differential equation of deflection curve under assumption of small deflections ($w'^2 \ll 1$), we obtain the correct value of critical force F_{kr} at which equilibrium bifurcation occurs but we are not able to determine the deflections of the bar for the load values $F > F_{kr}$.

For the correct calculation of the critical force, it remains us to determine the plane in which the bending happens. It will be that plane for which the critical force F_{kr} is minimal, because in all other planes the bifurcation point would be achieved under a higher load. As the critical force F_{kr} is proportional to the moment of inertia of the cross section J ($F_{kr} = \pi^2 \frac{EJ}{l^2}$), it will be minimal for the lower of both of the principal moments of inertia ($J = J_2$). It means that the neutral axis of bending is identical with the axis with respect to which the minimal moment of inertia J_2 is achieved. The deflection occurs in the direction of the J_1 axis, because bending in other directions would not occur but under a higher loading force. Therefore the deflections occur in the direction of the lower of both lateral dimensions for the bar in the figure having rectangular cross section.

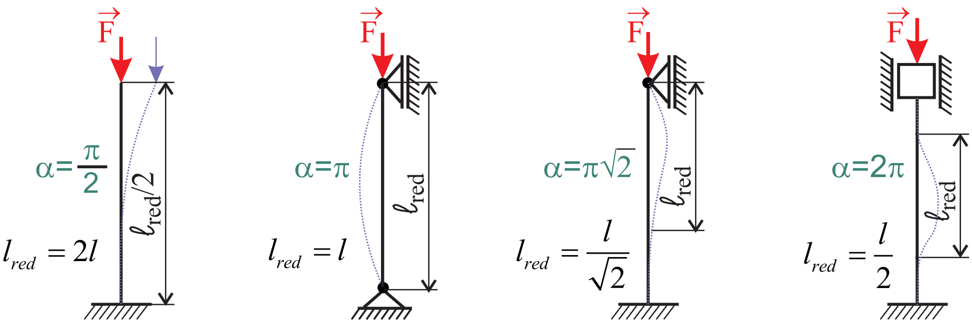


15.2. Critical force of buckling at a supported bar

Till now, we analysed the simplest case - a free bar loaded by two isolated forces being in static equilibrium on a common line of action. The following relation for the critical force of buckling F_{kr} was derived for a supported bar (e.g. in [1])

$$\boxed{F_{kr} = \alpha^2 \frac{EJ_2}{l^2}} \quad \text{or} \quad \boxed{F_{kr} = \frac{\pi^2 EJ_2}{l_{red}^2}.}$$

The **parameter** α is determined by the supports of the bar (it holds $\alpha = \pi$ for a free bar), the reduced length l_{red} can be determined according the figure. The reduced length of the supported bar equals to the length of a free bar having the same critical force value as the analysed supported bar. As the bending moment equals zero in the ending points of a free bar even in the deformed state, the reduced length equals to the lowest distance between two points with zero bending moment on the deflection curve of the analysed supported bar.



The derived relations hold for an ideal bar under ideal loads; the safety factor for the limit state of buckling of the bar can be calculated using the formula

$$k_V = \frac{F_{kr}}{F}.$$

If deviations from the above assumptions of an ideal column are negligible, this value can be used also for safety factor calculation of a real column. However, higher safety factor values should be chosen, usually $k_V \in \langle 3; 5\rangle$. On the other side, if the deviations from the above assumptions of an ideal column are substantial, a continuous increase of deflections occurs since the very beginning of the loading process; it is a combination of compression and flection and the limit state of buckling cannot occur at all.

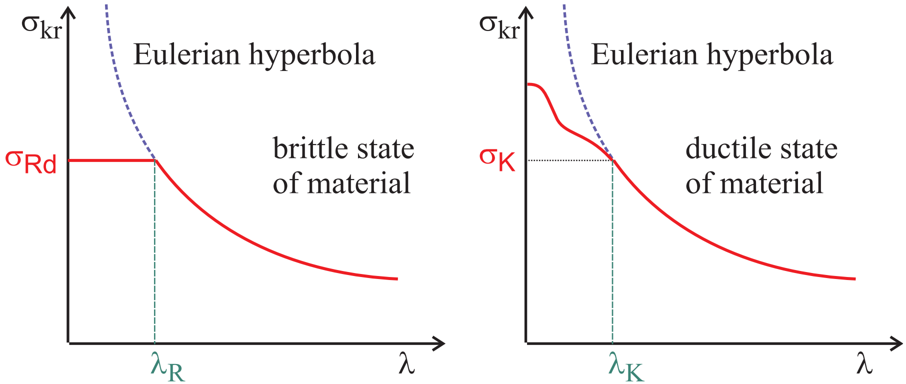
15.3. Compressive load of a column of a real material

Till now we assumed that the material behaviour is linear elastic without any limitations, so that neither plastic deformations nor fracture occur. Real materials are either ductile with a pronounced plastic deformation when yield stress σ_K is achieved, or brittle, at which a brittle fracture suddenly occurs if $|\sigma| = \sigma_{Rd}$. In the equilibrium bifurcation point, the stress achieves the magnitude

brittle fracture

$$\sigma_{kr} = \frac{|N|}{S} = \frac{F_{kr}}{S} = \alpha^2 \frac{EJ_2}{l^2 S} = \alpha^2 \frac{E}{\lambda^2}, \quad \text{where } \lambda = \frac{l}{\sqrt{\frac{J_2}{S}}} = \frac{l}{i} \quad \text{is the so called slenderness ratio.}$$

The quantity $i = \sqrt{\frac{J_2}{S}}$ is called radius of gyration and it serves for comparison of the slenderness of columns with different shapes of cross-sections.



The graph of the dependence of the compression stress σ_{kr} in the equilibrium bifurcation point on the slenderness ratio λ of the column is a hyperbola of higher order (Eulerian hyperbola). The derived relation for the critical force of buckling holds only in the case that the critical stress σ_{kr} is less than the limit of the linear material behaviour. The critical slenderness ratio of the column corresponds to the point in which both of these stress values are equal. The critical slenderness ratio is denoted λ_R or λ_K for brittle and ductile material, respectively.

a) Brittle material:
The buckling can occur if $\sigma_{Rd} > \sigma_{kr} = \alpha^2 \frac{E}{\lambda^2}$, i.e. for slenderness ratio of the column $\lambda > \alpha\sqrt{\frac{E}{\sigma_{Rd}}} = \lambda_R$. For $\lambda < \lambda_R$ the failure of the column by brittle fracture comes into being.

b) Ductile material:
The elastic buckling can occur if $\sigma_K > \sigma_{kr} = \alpha^2 \frac{E}{\lambda^2}$, i.e. for slenderness ratio of the column $\lambda > \alpha\sqrt{\frac{E}{\sigma_K}} = \lambda_K$. For $\lambda < \lambda_K$ the limit state of elasticity is achieved before the buckling can occur. Also in this case buckling can come into being if the load continues to increase but it is plastic buckling already and the derived relations do not hold any more.

When solving tasks with columns we must decide which of the possible limit states comes into being as the first. Let's present an example of a column made of material in ductile state. In common structures we allow neither plastic deformations nor buckling. Then the following statements are valid: Example 702

a) for $\lambda > \lambda_K \;\Rightarrow\;$ the limit state of buckling is decisive, $F_{kr} = \alpha^2 \frac{EJ_2}{l^2}$ and the safety factor corresponding to the buckling will be $k_v = \frac{F_{kr}}{F}$,

b) pro $\lambda < \lambda_K \;\Rightarrow\;$ the limit state of elasticity is decisive, and the safety factor corresponding to the limit state of elasticity will be $k_K = \frac{\sigma_K}{\sigma_{\max}}$.

Problem 701

15.4. Examples and problems

Examples

Problem 702

Problems

Problem 701
