12. Simple torsion

12.1. Definition

Simple torsion is loading of a straight prismatic bar, if

- bar assumptions are satisfied,
- cross sections do not deform, they only mutually rotate around the bar centreline,
- only non-zero component of the inner resultants is torsion moment (torque) M_k,
- deformations of the bar are not significant from the viewpoint of element equilibrium,
- cross section is axisymmetric (circle or annulus).

bar assumptions

Notes to the definition

In the beginning of development of the theory of elasticity of bars, the cross sections were not limited to the axisymmetric ones. The increasing resolution level (development of measuring equipments), however, brought findings that the cross sections remain planar (with a sufficient accuracy) only if the cross sections are axisymmetric; in contrary, a significant warping occurs at all the other cross section shapes. Therefore the formulas derived below are not valid for non-axisymmetric cross sections; they can be solved

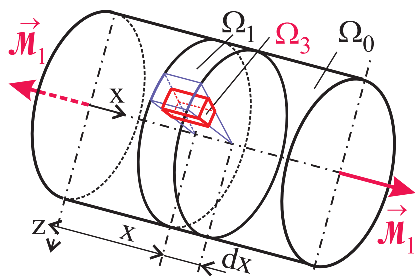
- using methods of general theory of elasticity (bars with cross sections having the following shapes: equilateral triangle, ellipse, circle with eccentric circular hole),
- using analytical methods (rectangle, square),
- using the finite element method (any shapes).

In contrast to the simple tension theory where we distinguished between tension and compression pursuant to the orientation (sign) of the normal force N, the orientation of the torsion moment is not significant; the body made of an isotropic material behaves in the same way for both orientations of torques.

12.2. Geometrical relations

Since we deal with the bars of axisymmetric cross sections only, we can advantageously use a cylindrical coordinate system with coordinates x, r and φ in axial, radial and circumferential directions, respectively. It can be stated on the deformation of the onefold (Ω_1) and threefold (Ω_3) **infinitesimal elements** during the loading process:

- the distance $\mathrm{d}x$ of the cross sections ψ_1 and ψ_2 remains preserved, so that the length strain in the direction of the bar centreline $\varepsilon_x = 0$ equals zero (under assumption of small strains),
- dimensions of the cross sections do not change so that the length strains in radial $(\varepsilon_r = 0)$ and circumferential $(\varepsilon_\varphi = 0)$ directions equal zero,
- since the cross sections remain planar, the right angle between the radial and axial directions remains unchanged (therefore $\gamma_{xr} = 0$),



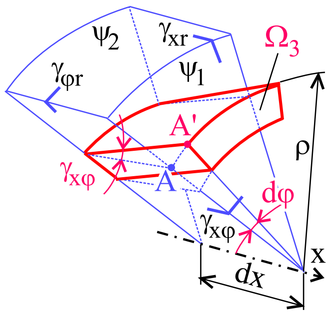
element

strain

- angular strains $\gamma_{\varphi r}$ are zero in a consequence of the axisymmetric character of deformation,
- faces of the element Ω_3 rotate mutually by the angle $\mathrm{d}\varphi$ what results in the non-zero angular strain $\gamma_{x\varphi}$; the distribution of this strain throughout the cross section can be obtained by expressing the displacement $\widehat{AA'}$ of a point A in a general cylindrical section defined by the radius ρ : $\widehat{AA'} = \mathrm{d}x\gamma_{x\varphi}$ what can be expressed dually by the geometrical parameters lying in the cross section: $\widehat{AA'} = \rho \mathrm{d}\varphi$.

$$\gamma_{x\varphi}\mathrm{d}x = \rho\mathrm{d}\varphi \quad \Rightarrow \quad \gamma_{x\varphi} = \rho\frac{\mathrm{d}\varphi}{\mathrm{d}x} \quad \Rightarrow \quad \gamma_{x\varphi} = \gamma = \rho\vartheta,$$

where $\vartheta = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x}$ is the **relative twisting angle** being constant for the given cross section.



The angular strain $\gamma_{x\varphi} = \gamma$ is the only non-zero component of the strain tensor in the case of the simple torsion; the distribution of this strain throughout the cross section is linear with the zero value on the bar centreline ($\gamma = \rho\vartheta$).

A specific deformation state occurs in the bar, denoted as **shear strain state**;

it can be described by the strain tensor in following shape $T_\varepsilon = \begin{pmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. T_ε

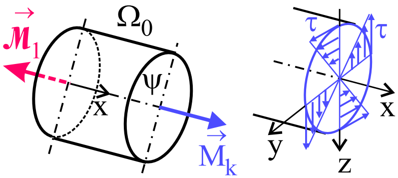
12.3. Stress distribution in the cross section

The stress distribution throughout the cross section can be obtained using constitutive relations valid for the Hookean (homogeneous, linear elastic) material in the shape $\sigma = E\varepsilon$ in the case of the uniaxial stress state and $\tau = G\gamma$ in the case of the shear stress state.

geometrical relations

It holds for simple torsion:

$$\begin{aligned} \varepsilon_x = \varepsilon_r = \varepsilon_\varphi = 0 \quad &\Rightarrow \quad \sigma = 0, \\ \gamma_{xr} = \gamma_{\varphi r} = 0 \quad &\Rightarrow \quad \tau_{xr} = \tau_{\varphi r} = 0, \\ \gamma_{x\varphi} = \gamma \neq 0 \quad &\Rightarrow \quad \tau_{x\varphi}(\rho) = \tau(\rho) = G\gamma = G\rho\vartheta. \end{aligned}$$

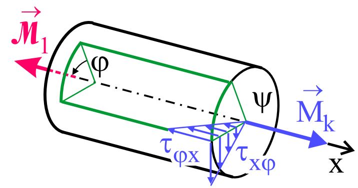


Under simple torsion, shear stresses come into existence in the cross section; the distribution of these stresses throughout the cross section is linear with zero value on the bar centreline. Normal stresses equal zero.

stress state

shear stress equality

The stress state in a point of the body, which is determined by an only one shear stress component, is denoted as **shear stress state**.
A shear stress $\tau_{x\varphi}$ in a section containing the bar centreline corresponds to the shear stress $\tau_{\varphi x}$ in the cross section ψ; both of these stresses are equal in magnitude (principle of shear stress equality):

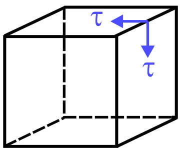


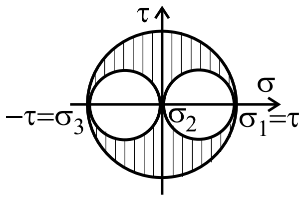
$$\tau_{x\varphi} = \tau_{\varphi x} = \tau$$

stress tensor

Mohr's plane

The shear stress state can be expressed in the following matrix form of the stress tensor, or represented on the threefold elementary brick or in the Mohr's plane.



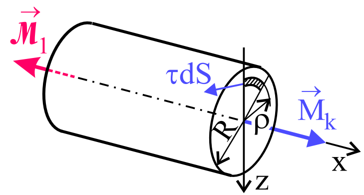


$$T_\sigma = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

12.4. Dependence between inner resultants and stresses

static equivalence

There is an only one applicable equation of static equivalence between the system of inner elementary plane forces in the cross section represented by the shear stress τ and their resultant $\vec{M}_k$; we use this equation to determine the dependency of shear stresses in the cross section on the inner resultants and on the geometrical characteristics of the cross section:



$$\sum M_x : \quad M_k = \int_\psi \mathrm{d}M_x = \int_\psi \rho\tau\mathrm{d}S = \int_\psi G\vartheta\rho^2\mathrm{d}S = G\vartheta\int_\psi \rho^2\mathrm{d}S = G\vartheta J_P,$$

where J_P is the polar quadratic moment of the cross section.

J_P

By simple manipulations we obtain further relations from this equation:

geometrical relations

- relative twisting angle $\quad \vartheta \quad = \quad \dfrac{M_k}{GJ_P}$
- angular strain $\quad \gamma \quad = \quad \rho\vartheta = \dfrac{M_k}{GJ_P}\rho$

stress

- shear stress $\quad \tau(\rho) \quad = \quad G\gamma \quad \Rightarrow \quad \boxed{\tau(\rho) = \dfrac{M_k}{J_P}\rho}$

12.5. Extreme stress

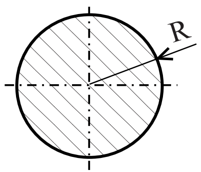
The shear stress $\tau(\rho) = \frac{M_k}{J_P}\rho$ will be maximal in those locations of the cross section where the radius is maximal, i.e. on the outline of the cross section: $\tau(\rho)$

$$\tau_{ex} = \frac{M_k}{J_P}\rho_{ex} = \frac{M_k}{\frac{J_P}{\rho}_{ex}} = \frac{M_k}{W_k},$$

where we introduced the **torsion modulus of the cross section** $W_k = \frac{J_P}{\rho_{ex}}$.

The torsion modulus of the cross section

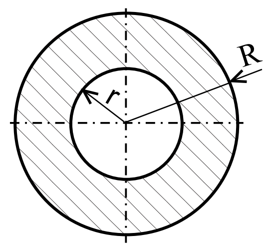
– for a circular section



$$W_k = \frac{J_P}{\rho_{ex}} = \frac{J_P}{R} = \frac{\frac{\pi R^4}{2}}{R} = \frac{\pi R^3}{2} = \frac{\pi D^3}{16}$$

– for an annular section

quadratic moment



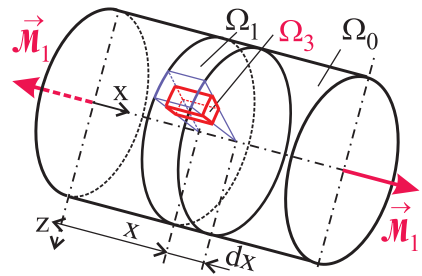
$$W_k = \frac{\frac{\pi}{2}(R^4 - r^4)}{R} = \frac{\pi R^3}{2}\left[1 - \left(\frac{r}{R}\right)^4\right] = \frac{\pi D^3}{16}\left[1 - \left(\frac{d}{D}\right)^4\right]$$

Warning! W_k **is not additive**, in contrast to the quadratic moments (there is $\rho_{ex} = R$ in the denominator in all cases, therefore the modulus of the small circle W_{k2} cannot be subtracted from the modulus of the large circle W_{k1}).

12.6. Strain energy

Under assumptions of the linear theory of elasticity, all the deformation work is transformed into the reversible strain energy $A = W$. linear theory

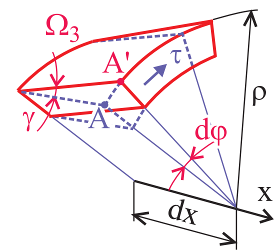
During the rotation of the the threefold elementary element Ω_3 of the length dx by angle dφ, the inner elementary shear force $\tau \mathrm{d}S\vec{\jmath}$ acting on element Ω_3 does the work element



geometrical relations

$$A_{\tau \mathrm{d}S} = \frac{1}{2}\tau \mathrm{d}S\widehat{AA'} = \frac{1}{2}\tau \mathrm{d}S\gamma \mathrm{d}x.$$

The strain energy W_{Ω_3} of the element Ω_3 (after substitution of the constitutive relation $\gamma = \frac{\tau}{G}$) and the strain energy density Λ (related to a unit volume dSdx) equal: Hooke's law



$$W_{\Omega_3} = A_{\tau \mathrm{d}S} = \frac{\tau^2}{2G}\mathrm{d}S\mathrm{d}x,$$

$$\Lambda = \frac{W_{\Omega_3}}{\mathrm{d}S\mathrm{d}x} = \frac{\tau^2}{2G} \qquad \Rightarrow \qquad \boxed{\Lambda = \frac{1}{2}\tau\gamma = \frac{1}{2}G\gamma^2.}$$

Note:
The formula for the strain energy density is analogical to the formula derived for the simple tension. tension

The formulas hold generally for any shear stress state. The strain energy W_{Ω_1} of a onefold elementary element can then be evaluated by integration of the following relation throughout the cross section ψ (and by substituting $\tau = \frac{M_k}{J_P}\rho$, $J_P = \iint\limits_{\psi} \rho^2 \mathrm{d}S$)

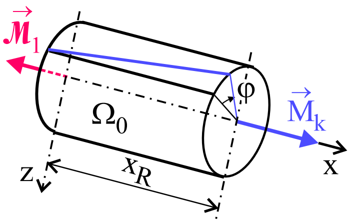
$$W_{\Omega_1} = \iint\limits_{\psi} W_{\Omega_3} = \iint\limits_{\psi} \frac{\tau^2}{2G} \mathrm{d}S\mathrm{d}x = \iint\limits_{\psi} \frac{M_k^2}{2GJ_P^2}\rho^2 \mathrm{d}x\mathrm{d}S = \frac{M_k^2}{2GJ_P^2}\mathrm{d}x \iint\limits_{\psi} \rho^2 \mathrm{d}S \; = \frac{M_k^2}{2GJ_P}\mathrm{d}x,$$

The total strain energy accumulated in a bar of length l is

$$\boxed{W(l) = \int\limits_0^l W_{\Omega_1} = \int\limits_0^l \frac{M_k^2}{2GJ_P}\mathrm{d}x.}$$

12.7. Expression for deformation characteristic of bar centreline

The deformation of the bar is described by the angle of mutual rotation (twisting) dφ of two adjacent sections ψ_1 and ψ_2 of the element Ω_1 $\vartheta(\varphi)$



$$\mathrm{d}\varphi = \vartheta \mathrm{d}x = \frac{M_k}{GJ_P}\mathrm{d}x.$$ $\vartheta(M_k)$

The rotation angle φ of the cross section cutting away the finite element Ω_0 is determined by the integral along the length of this element

$$\varphi(x_R) = \int_{x_m}^{x_R} \frac{M_k(x)}{GJ_P(x)}\mathrm{d}x,$$

where x_R is the coordinate of the gravitational center of the section in question, while x_m is the coordinate of the referential section (usually fixed, having therefore zero rotation angle).

If it holds $M_k(x)$ =const., $GJ_P(x)$ =const. in a certain part of the centreline and if we locate the origin of the used coordinate system to the gravitational center of the section with zero rotation angle ($x_m = 0$), then

$$\varphi(x_R) = \frac{M_k x_R}{GJ_P},$$ GJ_P is denoted as the **torsional stiffness of the cross section**.

12.8. Deformations of the cross section

Neither dimensions nor the shape of the cross sections do change under conditions of simple torsion. If this is the case, then the assumptions of the simple torsion would be bar assumptions violated and the above theory cannot be valid (e.g. warping-buckling of the cross section by a loss of shape stability in torsion of a thin-walled tube).

12.9. Solving problems concerning simple torsion of bars

12.9.1. Free bar

We derived the relations for stress, deformation parameter and strain energy valid for a bar under torsion if the bar assumptions are satisfied.
It holds for bars with **circular** or **annular** cross section:

$$\tau = \frac{M_k(x_R)}{J_P(x_R)}\rho; \quad \tau_{ex} = \frac{M_k(x_R)}{W_k(x_R)}; \quad \varphi(x_R) = \int_0^{x_R} \frac{M_k(x)}{GJ_P(x)}\mathrm{d}x; \quad W(l) = \int_0^{l} \frac{M_k^2(x)}{2GJ_P(x)}\mathrm{d}x.$$

bar assumptions

τ

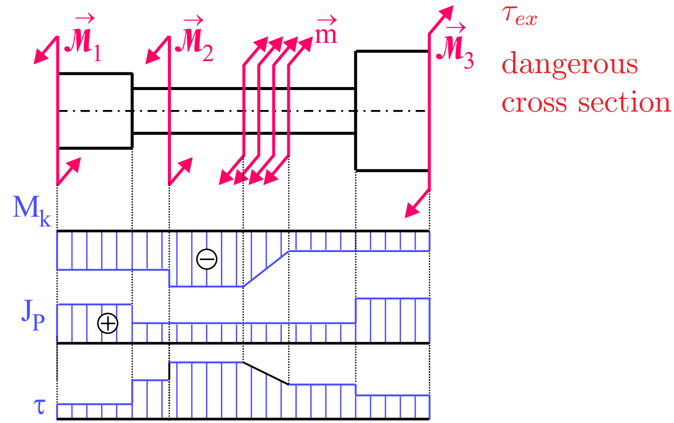
φ

W

tension

centreline

If $M_k(x)$, $S(x)$ or G change along the bar centreline (in the way, however, that does not violate substantially the assumptions of the simple loading), then the bar centreline must be divided into intervals (similar to the simple tension) in which each of the above quantities is expressed by a single (continuous) relation. The borders of the intervals are in those points of the centreline, where a change occurs in the functions describing the distribution of $G(x)$, $M_k(x)$ or $J_P(x)$ along the centreline.
The shear stress distribution in the cross section is linear with the extreme value on the outline in the case of simple torsion. Therefore all the outline points of the dangerous cross section are dangerous points (with the same extreme value of safety factor).



The rotation angle of the cross section can be calculated:

- using the derived relation for the twisting angle of the cross section, the gravitational center of which has coordinate x_R:

$$\varphi(x_R) = \int_0^{x_R} \frac{M_k(x)}{GJ_P(x)} \mathrm{d}x$$

twisting

- using Castigliano's theorem – the twisting angle φ_B of the point of action of a couple $\vec{\mathcal{M}}_B$ in the plane of its action equals

Castigliano's theorem

$$\varphi_B = \frac{\partial W}{\partial \mathcal{M}_B} = \int_0^{l} \frac{M_k(x)}{GJ_P(x)} \frac{\partial M_k(x)}{\partial \mathcal{M}_B} \mathrm{d}x.$$

Both of these relations are equivalent, the partial derivative $\dfrac{\partial M_k(x)}{\partial \mathcal{M}_B}$ usually equals ± 1, so that the results can differ in the sign only. The limit state of deformation is determined by the value of the twisting angle φ_M inadmissible in operation, the safety factor related to this limit state can be calculated using the formula $k_\varphi = \frac{\varphi_M}{\varphi_{\max}}$.

limit state of deformation

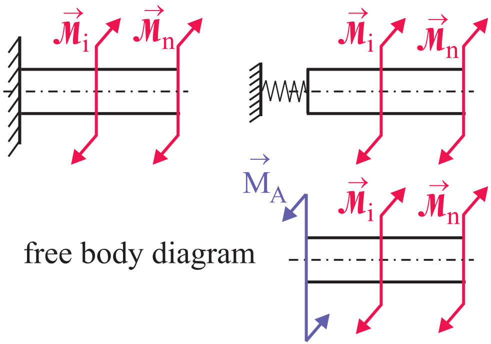
safety factor

The safety factor related to the yield limit can be calculated using the formula $k_K = \frac{\tau_K}{|\tau_{\max}|}$. The yield stress in tension σ_K cannot be used here as the limit value, shear yield stress must be used instead. In practice, this value is not measured but calculated from the relation $\tau_K = \frac{\sigma_K}{2}$ based on the Tresca's $(\max \tau)$ plasticity criterion.

12.9.2. Supported bar

Problem 501
redundancy

The bearing of a bar under torsion is statically determinate, if rotations of the cross section are restricted in one point of its centreline.



The free body diagram (for both of the presented bars loaded only by couples $\vec{\mathcal{M}}_i$ in parallel planes, the only non-zero component of support reactions is then the couple $\vec{M}_A$) shows that there is only one applicable equation of static equilibrium

$$\sum M_x = 0 : \qquad M_A - \sum_{i=1}^{n} \mathcal{M}_i = 0$$

$s = \mu - \nu = 1 - 1 = 0 \quad \Rightarrow$ statically determinate bearing.

All the other types of bearings of bars loaded in torsion are statically indeterminate.

The solutions to supported bars loaded in torsion can be carried out according the algorithm presented in the chapter 11.11.2. Supported beam. It must be, however, taken into account that

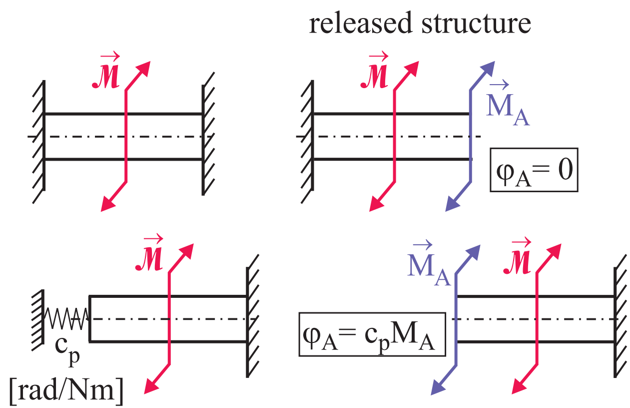
algorithm

Example 507

Problem 503

Problem 505

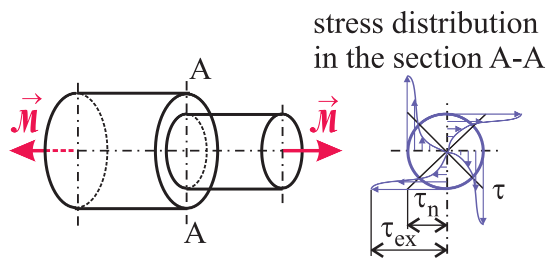
- the moment equation related to the x-axis $\sum M_x = 0$ is the only one applicable condition of static equilibrium,
- the compatibility equations are determined by rotation angles of the cross sections around the bar centreline in such a number of points that equals to the degree of redundancy. The compatibility equations (support deformation conditions) can be homogeneous, non-homogeneous or circumstantial.



Note:

If the bearing of the bar consists of both rigid and flexible supports, a rigid support should be preserved in released structure (then the body remains immovable as a whole) and non-homogeneous compatibility equations should be formulated for flexible supports; otherwise the bearing of the released structure would not be immovable and the problem of how to distinguish between movements of the whole body and deformations should be solved additionally. The compatibility equations are dependent on the load only, there is usually no influence of temperature changes and production inaccuracies. If a significant change in temperature or length inaccuracy occurs at a statically indeterminate bar loaded in torsion, a non-zero normal force occurs and the onefold loading of the bar is changed in a combined loading (torsion + tension or compression).

– Also at bars under torsion the problems of notches must be taken into account, because stress and strain concentrations occur there. The extreme stress value in the notch root can be calculated using the formula $\tau_{ex} = \tau_n$,



Problem 502
Problem 504
Problem 506

where

– α is a stress concentration factor evaluated from the graphs (nomograms) that have been based on calculations (finite element method) or on experiments (photoelasticimetry) for various shapes of notches,
– τ_n is the nominal stress in the notch location calculated using the theory of simple torsion.

α graphs

12.10. Examples and problems

Examples

Problem 507

Problems

Problem 501 Problem 502 Problem 503 Problem 504 Problem 505

Problem 506