17. Fundamentals of theory of limit states

At the beginning of this chapter, let's try to define the objectives of the theory of limit states. Most students, and engineers as well, identify all the mechanics of materials with the stress-strain analysis, i.e. calculation of stresses and strains in the bodies under load. The extreme calculated stresses (or deformations) can then be compared with the corresponding limiting values (yield stress, strength, ultimate stress, fatigue strength etc.), so where is the need of the scientific branch called theory of limit states?

We will show the reasons using an example of multiaxial state of stress. The simple procedure described above is namely sufficient only in the case that the state of stress (if the stress is the quantity decisive for the occurrence of the limit state in question, e.g. for plastic deformation or fracture) is defined only by a single non-zero component of the stress tensor T_σ. In the case of a uniaxial or shear (in plane) stress states, the evaluation of the risk of the limit state of elasticity is really as simple; in fact, students in the bachelors degree have not exercized any more complex stress states in their practical computations.

uniaxial stress state

shear stress state

However, the risk evaluation of the limit state is not so easy in the case of a more complex stress state, e.g. when there is a **bar-type state of stress** (defined by normal σ and shear τ components of stresses in the bar cross section) in the body in question. Try to answer the following question, very easy at first sight, concerning the illuminating example below: which of the stress states 1 and 2 (defined by the stress tensors T_{σ_1} and T_{σ_2} in the dangerous point of the body) is more dangerous, i.e. in which of them there is a higher risk of failure (loss of the body functionality)?

bar stress state

$$T_{\sigma_1} = \begin{pmatrix} 50 & 50 & 0 \\ 50 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_{\sigma_2} = \begin{pmatrix} 70 & 40 & 0 \\ 40 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The answer is not unambiguous, it depends on the type of material behaviour. The influence of the normal stress is higher at a brittle material, so that the risk of failure (brittle fracture) will be higher in the case 2, while the influence of the shear stress component will be higher at a ductile material (i.e. the stress state T_{σ_1} will bring a higher risk of plastic deformations of the body). The evaluation of a general stress state defined by six non-zero components of the stress tensor is even more complex; some of the stress components can increase, some others can decrease or remain unchanged between any two of the operational states of the body. E.g., if you should to judge the right front wheel axle of a vehicle, what of the possible operational states is the most dangerous, breaking, going through a sharp left curve or passing a pot-hole in the roadway? In all of these examples, the load of the wheel, axis or all the vehicle gear is quite different. Even the trivial question whether the transition from the above stress state 1 to the stress state 2

stress state

can be called loading or unloading is not easy to be answered. The generally valid answer can be formulated as follows:

The process of change of the stress-strain state in the body can be called loading, if there is a higher risk of a certain limit state (failure) in the final stress state than it was in the initial state. An inverse process is called unloading.

It is evident from the above facts that it is necessary to find a procedure of how to evaluate the risk of limit states (failure) if the decisive quantity shows a tensor character, i.e. it is defined by more independent components. Formulation of such procedures (so called limit state criteria) is the objective of this chapter and of the theory of limit states as a scientific branch.

17.1. Factor of safety

The ability of the structure to perform the intended operational functions under normal and some extraordinary (e.g. pressure test of a vessel) conditions is called **reliability of the structure**. The reliability is required to be quantified, i .e. we need to e valuate how large the changes of the quantities influencing the limit state occurrence can be until a failure occurs. As any of the measured or calculated quantities is stochastic, we cannot allow states near the limit state in the operation of the structure; there must be a margin called **safety**. To evaluate this safety, we need to find a physical quantity decisive for the limit state occurrence (e.g. normal, shear or reduced stresses, force, deflection, number of load cycles etc.). For this quantity, the **factor of safety** (called more **precisely** simple factor **of safety**) relating to the limit state in question is defined by the following general relation:

$$\boxed{k = \frac{\alpha_M}{\alpha_P},}$$

where α_M is the limiting value and α_P is the operational value of the decisive quantity α. In practice, the value of **factor of safety** must be $k > 1$. If it holds $k = 1$, the corresponding limit state occurs. For the particular limit states we can define

the **factor of safety** against

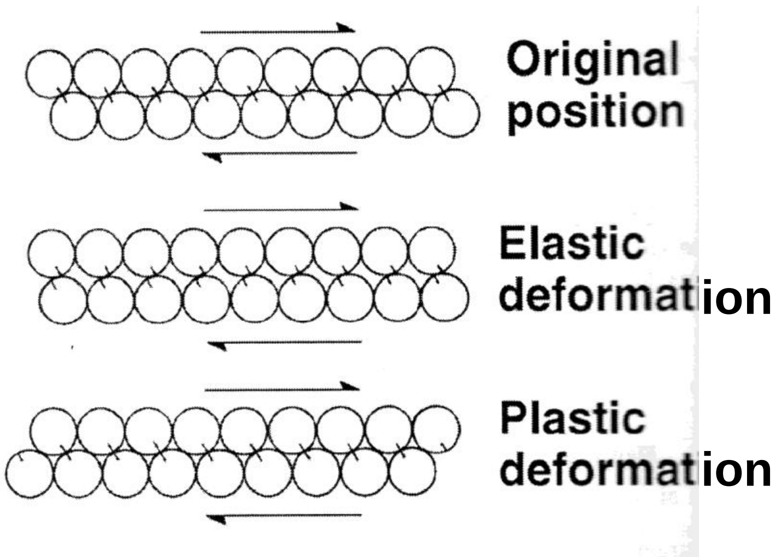
- limit state of deformation $k_D = \frac{u_{\text{limiting}}}{u_{\max}}$
- limit state of elasticity $k_K = \frac{\textit{yield stress}}{\textit{working stress}} = \frac{\sigma_K}{\sigma_{max}}$
- limit state of brittle fracture $k_R = \frac{\textit{strength in tension}}{\textit{working stress}} = \frac{\sigma_{Rt}}{\sigma_{max}}$

The above relations are valid if and only if the limit state is unambiguously defined by a single component of the decisive quantity (i.e for example in the case of a uniaxial stress state).

uniaxial stress state

To evaluate the factor of safety (**risk of failure**), we need some **failure criteria** – theory of limit states (failures).

Plasticity criteria (Tresca, Mises) are based on shear stresses (explanation in the figure below), while fracture (crack propagation) is more dependent on normal components of stresses.



17.2. Limit state of elasticity

Till now, we dealt with a *bar*, i.e. a model body loaded in tension (compression), torsion or flection. In the simple tension (compression) or flection, a uniaxial stress state occured in the bar, while **pure** shear occured in the bar under torsion. We solved stresses and deformations for these types of loads and we met limit states of **deformation and elasticity** (**yield**) when solving these tasks.

bar

tension

torsion

flection

We calculated the **factor of safety** against **yield** using the relations

$$\text{for tension and flection} \quad k_K = \frac{\sigma_K}{\sigma_{\max}}, \qquad \text{for torsion} \quad k_K = \frac{\tau_K}{\tau_{\max}} = \frac{\sigma_K}{2\tau_{\max}}.$$

The combined load of bars requires a description of limit states under conditions of **bar-type** state **of stress** (a particular case of biaxial stress state), other models (analytical or numerical) can result in even more complex stress states in dangerous points. The simplest level of description of the limit state of elasticity requires under conditions of a multiaxial stress state:

combine loading

bar stress state

- monotonously increasing loads (plasticity **criteria** do not hold for cyclic loa-ding),
- isotropic material from the viewpoint of yield stress (plasticity **criteria** do not depend on the directions of the stresses),
- uniparametric limit state (the limit state of elasticity is described by a single material characteristic, yield stress σ_K, which has the same magnitude in tension and in compression).

If we need to judge the risk of the limit state of elasticity (in the case of a multiaxial stress state), we need to formulate a general **plasticity criterion** (i.e. a mathematical description of the limit state of elasticity), and to know the limit value corresponding to this limit state (i.e. the **yield stress** – a material characteristic).

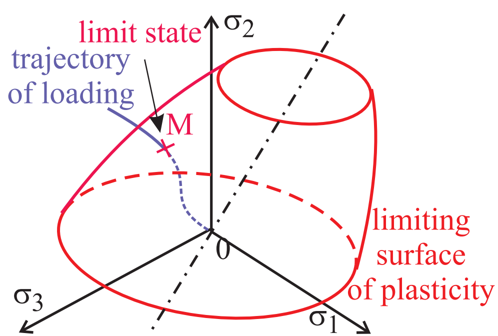
In the case of a **uniaxial stress state**, the relation $\sigma = \sigma_K$ can be denoted as the plasticity criterion; it can be expressed in the following general form:

$$F(\sigma) = \sigma_K, \qquad \text{where } F \text{ is a function of a single variable } \sigma \text{ in this case.}$$

The plasticity criterion for a **triaxial stress state** must be a function of the stress tensor T_σ, i.e. a function of the six independent stress components

$$F(T_\sigma) = F(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}) = \sigma_K.$$

It is advantageous to represent the plasticity criterion in the so called **Haigh space**; the coordinate axes of this space are identical with the principal axes of the stress state. In this space, the plasticity criterion is represented by a **surface of plasticity**, the loading process is represented by a curve – **trajectory of loading**. The limit state of elasticity occurs when the trajectory of loading intersects the limiting surface of plasticity (plasticity envelope).



Comprehensive experiments carried out during dozens of years resulted in the conclusion that a shear stress $|\tau_{\rho_K}|$ in a certain section ρ_K is the quantity decisive for the occurrence of the limit state of elasticity; the plasticity criterion can be then formulated as follows:

$$F(|\tau_{\rho_K}|) = M_K \qquad \text{(where } M_K \text{ is a material characteristic).}$$

A most simple function F which can be used in practice as plasticity criterion is a **linear** function; the corresponding plasticity criterion can be expressed in the form

$$F(|\tau_{\rho_K}|) = |\tau_{\rho_K}| = \tau_{MK}, \qquad \text{where } \tau_{MK} \text{ is a material characteristic.}$$

The section ρ_K was chosen on the basis of experimental experience; we can obtain various plasticity criteria in dependence on the choice of the relevant section.

17.2.1. Tresca's plasticity criterion (max τ)

The plasticity criterion based on the maximum shear stress assumes the section in which the maximum shear stress $\tau_{\max}$ acts to be the decisive section ρ_K; therefore the criterion can be formulated in the form

$$\boxed{\tau_{\max} = \tau_{MK}}$$

The limit state of elasticity comes into existence under conditions of monotonous loading of material in basic structural state (with the beginnning of loading in a stress-free state) if the maximum shear stress reaches its limiting value τ_{MK} which is a material characteristic.

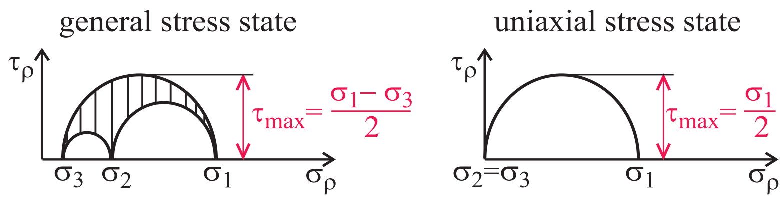
For a general stress state:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \tau_{MK}$$

For a uniaxial stress state:

$$\tau_{\max} = \frac{\sigma_1}{2} = \frac{\sigma_K}{2} = \tau_{MK},$$

because $\sigma_2 = \sigma_3 = 0$ and in the limit state of elasticity $\sigma_1 = \sigma_K$.



Our objective is to judge the risk of occurrence of the limit state under conditions of multiaxial stress states on the basis of experiments carried out in uniaxial stress state only (tension test); by comparing both types of stress states we obtain:

$$\tau_{MK} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_K}{2} \qquad \Rightarrow \qquad \boxed{\sigma_1 - \sigma_3 = \sigma_K}$$

By introducing the reduced stress

$$\boxed{\sigma_{red} = \sigma_1 - \sigma_3\,,}$$

we can obtain the form similar to the uniaxial stress state $\sigma_{red} = \sigma_K$ and we can calculate the **factor of safety** using the following formula

$$\boxed{k_K = \frac{\sigma_K}{\sigma_{red}}}\,.$$

Reduced stress σ_{red}
is a fictitious value of a uniaxial tensional stress giving the equal **factor of safety** against the *judged limit state* with the multiaxial stress state in question.

judged limit state

Note: the reduced stress is also called equivalent stress or stress intensity.

Evaluation of **risk of failure** using the reduced stress is then the same as in the case of the uniaxial stress state:

$\sigma_{red} < \sigma_K$	–	material is in elastic state,
$\sigma_{red} = \sigma_K$	–	the limit state of elasticity is reached,
$\sigma_{red} > \sigma_K$	–	material is in plastic state in the point in question.

The above general form of the Tresca's plasticity criterion max τ is valid for any stress state, it is however necessary to calculate all the three principal stresses. For some particular types of stress states (uniaxial stress state in bars loaded in tension, compression or flection, **pure shear stress** in bars loaded in torsion, **bar-type state of stress** in bars under combined load), the Tresca's plasticity criterion max τ or the formula for **reduced stress can** be simplified in the following forms:

principal stress

tension

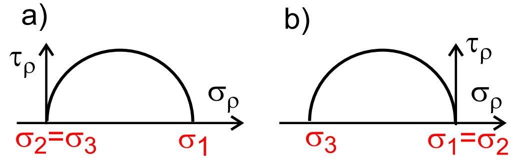
flection

torsion

1) Uniaxial state of stress

a) tensional

$$\sigma_1 = \sigma > 0, \quad \sigma_2 = \sigma_3 = 0 \;\Rightarrow\; \boxed{\sigma = \sigma_K}$$



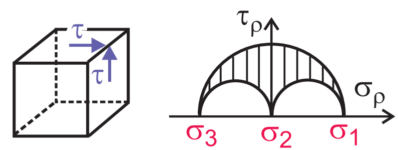
uniaxial stress state

b) compressional $\quad \sigma_3 = \sigma < 0, \quad \sigma_1 = \sigma_2 = 0 \;\Rightarrow\; \boxed{|\sigma| = \sigma_K}$

$$\boxed{\sigma_{red} = |\sigma|}$$

2) Pure shear stress

$$\sigma_1 = -\sigma_3 = \tau \qquad \sigma_2 = 0$$



shear stress state

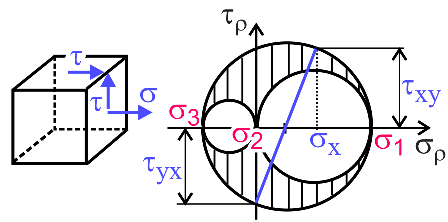
$$\sigma_1 - \sigma_3 = \tau - (-\tau) = \sigma_K \;\Rightarrow\; 2\tau = \sigma_K$$

$$\boxed{\sigma_{red} = 2\tau}$$

in the limit state of elasticity $\tau = \tau_K \;\Rightarrow\; \boxed{\tau_K = \frac{\sigma_K}{2}}$ ($\tau_K \ldots$ **yield shear stress**)

3) Bar-type state of stress

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \quad \sigma_2 = 0 \quad \sigma_3 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$



bar-type state of stress

By substitution in the plasticity criterion we obtain

$$\sigma_1 - \sigma_3 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} - \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \boxed{\sqrt{\sigma^2 + 4\tau^2} = \sigma_K} \qquad \boxed{\sigma_{red} = \sqrt{\sigma^2 + 4\tau^2}}$$

17.2.2. Mises' plasticity criterion (HMH)

Von Mises' plasticity criterion (the abbreviation HMH is based on initials of all the three authors of this criterion - Hencky, von Mises and Huber) assumes the octahedric section to be the decisive section ρ_K; therefore the criterion can be formulated in the form

octahedric plane

$$\boxed{|\tau_o| = \tau_{oK}}$$

The limit state of elasticity comes into existence under conditions of monotonous loading of material in basic structural state (with loading beginning in a stress-free state) if the shear stress in the octahedric plane reaches its limiting value τ_{oK} which is a material characteristic.

The shear stress in the octahedric plane (octahedric shear stress) can be calculated for a general stress state using the formula:

$$\tau_o = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}.$$

For the uniaxial stress state $(\sigma_1 = \sigma_K, \sigma_2 = \sigma_3 = 0)$ it holds:

τ_o

$$\tau_o = \frac{\sqrt{2}}{3}\sqrt{\sigma_K^2} = \tau_{oK} \quad \Rightarrow \quad \tau_{oK} = \frac{\sqrt{2}}{3}\sigma_K.$$

Our objective is to judge the risk of occurrence of the limit state under conditions of multiaxial stress states (for which material tests cannot be carried out) on the basis of experiments carried out in uniaxial stress state only, i.e. tension tests; yield stress is measured among other quantities in this tests. The Mises' criterion is based on comparison of octahedric shear stresses τ_o in both types of stress states. By comparison of the octahedric shear stresses in the general and uniaxial stress states we obtain

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}=\frac{\sqrt{2}}{3}\sigma_K$$

The Mises' plasticity criterion for the general stress state defined by principal stresses $\sigma_1, \sigma_2, \sigma_3$ can then be written in the following form:

$$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]}=\sigma_K$$

If we introduce the reduced stress in the way similar to Tresca's plasticity criterion presented above $\boxed{\sigma_{red}=\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]}}$,

the plasticity criterion can be simplified into the form $\sigma_{red}=\sigma_K$ and we can calculate the factor of safety using the same formula as in the case of Tresca's plasticity criterion $\boxed{k_K=\frac{\sigma_K}{\sigma_{red}}}$.

An important advantage of the Mises' plasticity criterion is the fact that it can be derived in the form based on the stress components in any general coordinate system, namely in the following form:

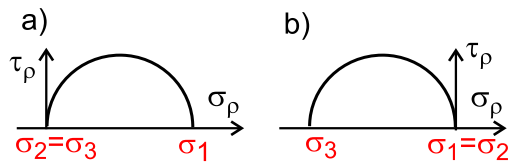
$$\sigma_{red}=\sqrt{\frac{1}{2}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_x-\sigma_z)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2\right)\right]}$$

1) Uniaxial state of stress

a) tensional

$$\sigma_1 = \sigma > 0, \quad \sigma_2 = \sigma_3 = 0 \ \Rightarrow$$

$$\sqrt{\frac{1}{2}\left(\sigma^2 + \sigma^2\right)} = \sigma_K \ \Rightarrow \quad \boxed{\sigma = \sigma_K}$$

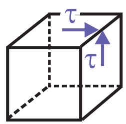


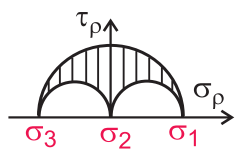
uniaxial stress state

b) compressional $\quad \sigma_3 = \sigma < 0, \quad \sigma_1 = \sigma_2 = 0 \ \Rightarrow \boxed{|\sigma| = \sigma_K}$

$$\boxed{\sigma_{red} = |\sigma|}$$

2) Pure shear stress





shear stress state

$$\sigma_1 = -\sigma_3 = \tau \qquad \sigma_2 = 0$$

$$\sigma_{red} = \sqrt{\frac{1}{2}\left[\tau^2 + \tau^2 + (2\tau)^2\right]} = \sigma_K \quad \Rightarrow \quad \sqrt{3}\tau = \sigma_K$$

in the limit state of elasticity $\tau = \tau_K \ \Rightarrow \boxed{\tau_K = \frac{\sigma_K}{\sqrt{3}}}$

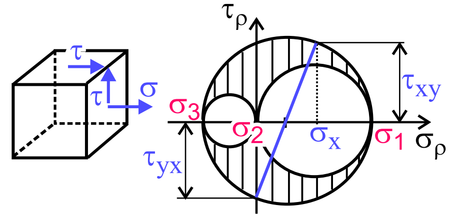
$$\boxed{\sigma_{red} = \sqrt{3}\tau}$$

($\tau_K \ldots$ yield shear stress based on Mises' plasticity criterion)

3) Bar-type state of stress

bar-type state of stress

$$\sigma_x = \sigma \neq 0;\ \tau_{xy} = \tau \neq 0;\ \sigma_y = \sigma_z = 0;\ \tau_{xz} = \tau_{yz} = 0.$$



These values can be directly substituted into the relation for the reduced stress in a general coordinate system and we obtain

$$\sigma_{red} = \sqrt{\frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right]} = \sqrt{\sigma^2 + 3\tau^2}$$

The Mises' plasticity condition valid for a bar-type state of stress is: $\sigma_K = \sqrt{\sigma^2 + 3\tau^2}$

Both of the presented plasticity criteria are equivalent in practical use. In analytical calculations, Tresca's plasticity condition max τ is often used because of its simpler form. Before using this condition, however, it is necessary to calculate the principal stresses and to order them in the decreasing sequence ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), because one of the principal stresses (σ_2) absents in the used formula. The formula expressing Mises' criterion is more complicated but this is no problem in computer solutions where it is used more frequently. Its advantage is that it was derived in the form based on the stress components in a general coordinate system so that it does not require knowledge of the principal stresses.

principal stress

17.3. Brittle fracture – failure of a body without any macroscopic crack

Brittle strength = a special case of brittle fracture with monotonically increasing load of the body.

- Brittle fracture: $\varepsilon < 0.001$
- Quasi-brittle
- Quasi-ductile
- Ductile fracture: $\varepsilon > 0.05$

Factors influencing the magnitude of plastic deformation until fracture (**brittle or ductile** material behaviour):

- Temperature – lower temperature means more brittle behaviour.
- Deformation speed – faster load means less plastic deformation, i.e. more brittle behaviour.
- Stress state – more triaxial stress means more brittle behaviour.
- Corrosion
- Radiation (x- or gamma-radiation)

How to avoid **brittle fracture**?

- Choice of material with transition temperature of brittle fracture being below the operating temperatures
- Production technology – without macroscopic cracks
- Avoid impact loading
- Reduce stress concentration in notches, corrosion and radiation

Summary of **experimental results** – internal factors influencing the occurrence of brittle fracture in a given material are as follows:

- Spherical part of the stress tensor (hydrostatic stress)
- Sign of principal stresses – positive values are more dangerous
- Occurrence of some small plastic deformation
- **Magnitude of shear stress τ_ρ and of normal stress σ_ρ in a certain characteristic section ρ.**

For a multiaxial stress state, the failure is described by **criteria of brittle fracture,** which can be valid if

- there is no initial macroscopic crack in the body
- the load is monotonically increasing
- initiation and propagation of the crack are instantaneously followed by fracture – fast fracture process
- the crack propagation in unstable and cannot be influenced by any changes in loads
- the stress state is homogeneous – otherwise an approximate validity only, because the conditions are changing during the crack propagation

Mathematical description of criteria of brittle fracture:

- ***Maximum principal stress criterion***

The occurrence of the brittle fracture under the above conditions is given by the value of maximum principal stress of the respective stress state:

$$\sigma_1 = R_m$$

where R_m means ultimate stress (strength) in tension.
This criterion is valid (corresponds to experimental results) only if all the three **principal stress values are positive** (multiaxial tension).

- ***Mohr's criterion***

The occurrence of the brittle fracture under the above conditions is given by the values of normal and shear stresses in the section, in which the maximum shear stress is acting. Mathematical formulation of this criterion can be expressed in the following shape:

$$\sigma_1 - \frac{R_m}{R_{mC}}\sigma_3 = R_m \qquad \text{or} \qquad \sigma_1 - \chi\sigma_3 = R_m$$

where R_m and R_{mC} mean ultimate stresses (strengths) in tension and in compression, respectively. Their ratio χ meets always the inequality $\chi < 1$.

This criterion is valid (corresponds to experimental results) only if **at least one of the three principal stress values is negative**.

- **MOS criterion**

This criterion is the only one, which is valid generally, independently of the stress state type and the signs of the stresses. It represents a combination of the previous two criteria and can be formulated mathematically as follows:

$$\max\left\{\sigma_1;\sigma_1-\chi\sigma_3\right\}=R_m$$

To avoid brittle fracture, a corresponding inequality must be met:

$$\max\left\{\sigma_1;\sigma_1-\chi\sigma_3\right\}<R_m$$

To quantify the margin of safety against the brittle fracture, reduced stress can be introduced (similarly to plasticity criteria) by the following formula:

$$\sigma_{redMOS}=\max\left\{\sigma_1;\sigma_1-\chi\sigma_3\right\}$$

The reduced stress (a simplification of stress tensor valid under the specified conditions only) can be applied for calculation of the simple factor of safety (FOS) under multiaxial stress states using the formula:

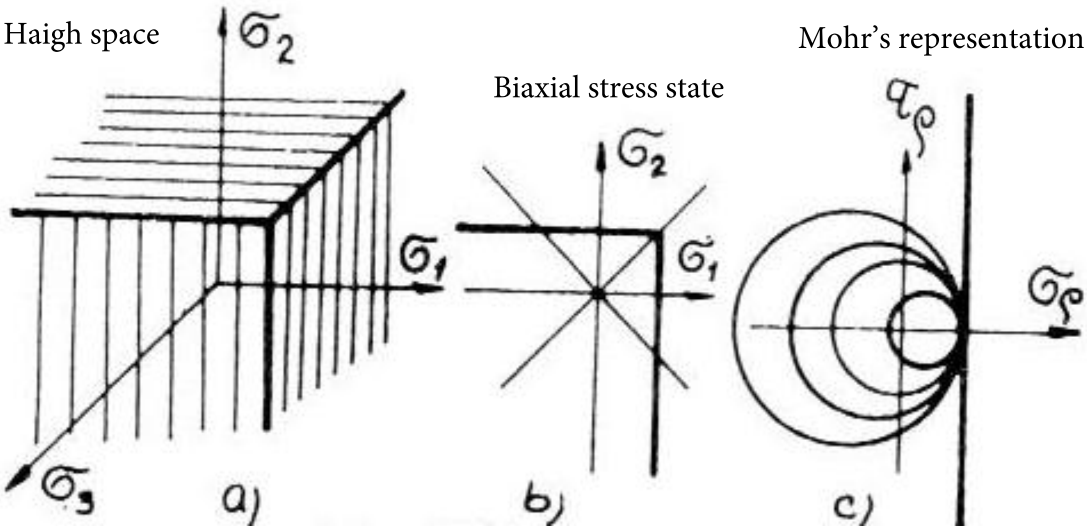
$$FOS=k_R=\frac{R_m}{\sigma_{redMOS}}$$

As materials in brittle state do not offer any margin of safety due to no plastic deformation (in opposite to materials in ductile state), the recommended range of the factor of safety against brittle fracture is much higher (3÷10).

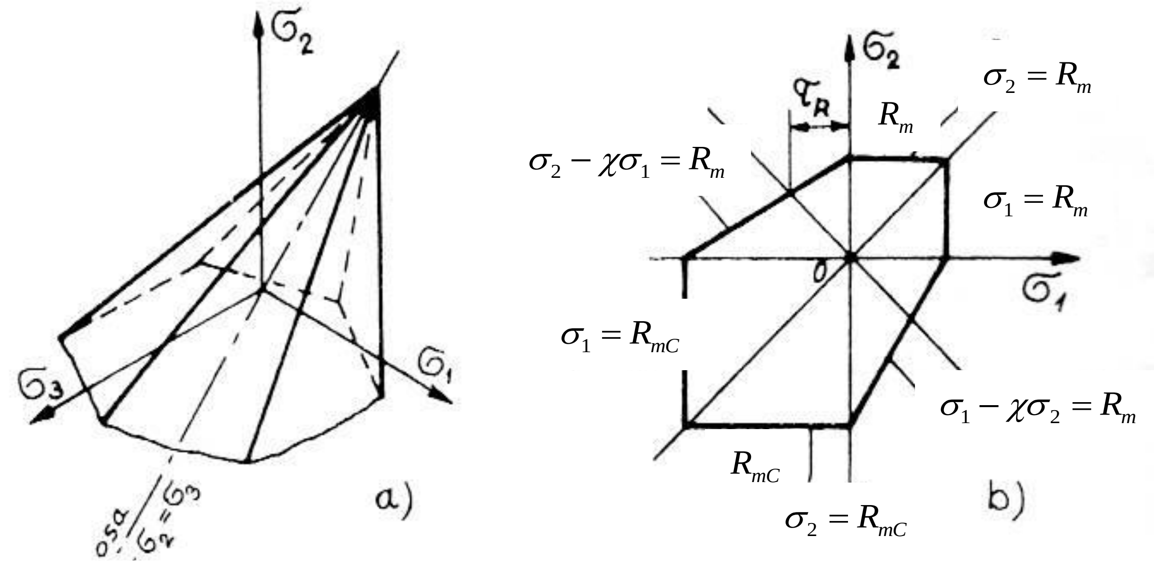
Note: *Reduced stress σ_{red} is a fictitious value of a uniaxial tension stress giving the same factor of safety against the judged limit state with the multiaxial stress state in question. As it simplifies the tensor into one numerical value only, it can be valid for a certain failure criterion only (and for one type of failure, particularly here for MOS criterion of brittle fracture) and it is necessary to distinguish between various types of reduced stresses.*

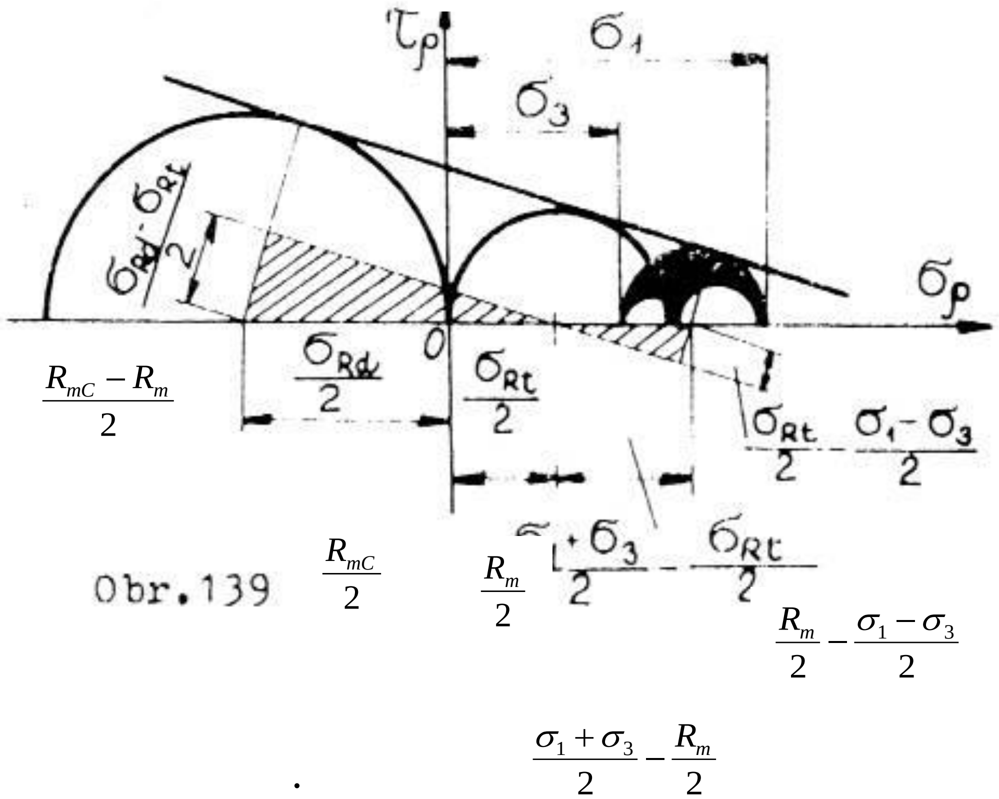
Graphical representation of the criteria of brittle fracture:

Maximum principal stress criterion:



Mohr's criterion:





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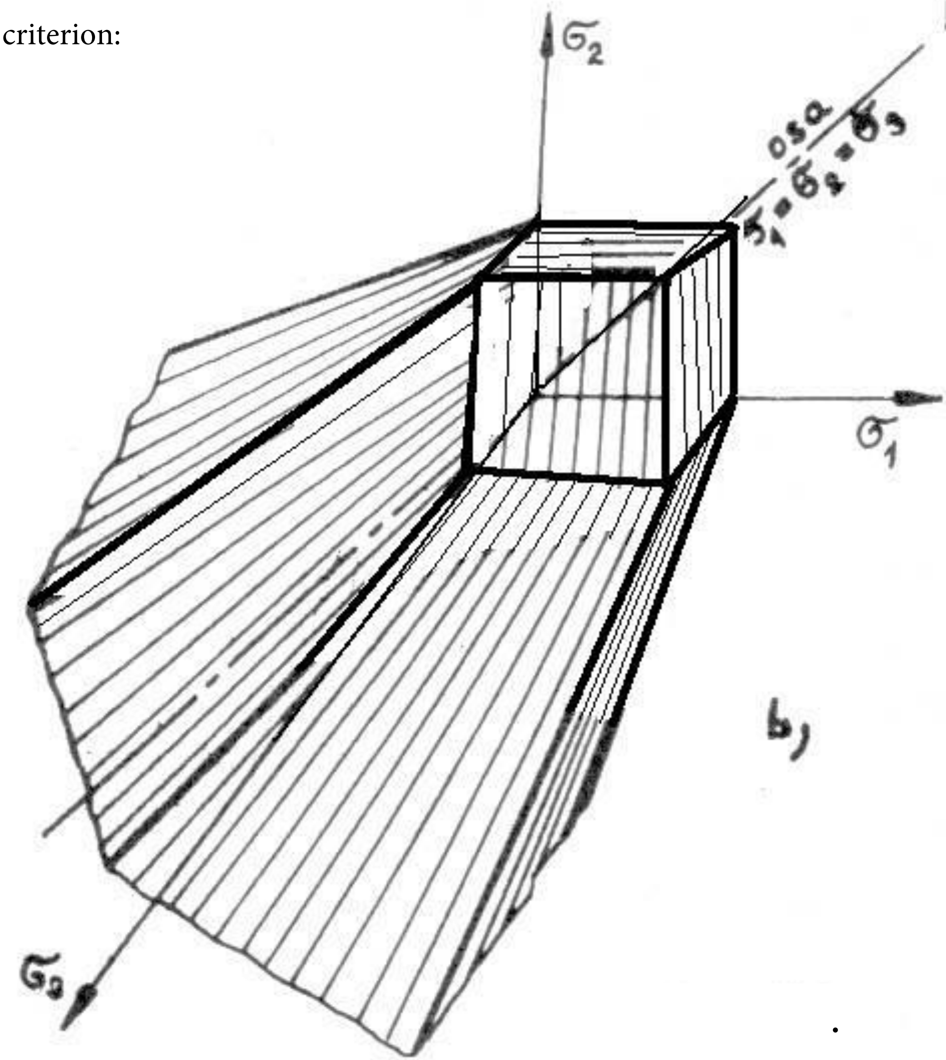
MOS criterion:



17.4. General and simple factor of safety

In the previous paragraphs, we introduced the **factor of safety** in the form

$$k_k = \frac{\sigma_K}{\sigma_{red}}$$

as a quantity which quantifies the safety against **yield stress** σ_K. However, the influence of the particular stress components on the reduced stress is different, so that this **factor of safety** can correctly evaluate the safety only if the increase of all the stress components during loading and overloading is mutually proportional. Such a way of loading is graphically (e.g. in Haigh stress space) represented by a straight line and it is called **simple** loading and overloading. Haigh space

The factor of safety valid for the simple loading is called simple factor of safety and it can be calculated using the reduced stress. If the stress components are not mutually proportional (e.g. the increase of torque is not proportional to the increase of bending moment and therefore also increase of shear stress τ is not proportional to the increase of normal stress σ), the reduced stress cannot be used; under these conditions, the general factor of safety should be evaluated which takes the trajectory of loading and overloading into account.