4. Stress state in a point of the body

Let's assume that the point C is the dangerous point of the body, that means point with maximum stress values and therefore with highest risk of failure; to avoid limit states of the body, the stresses in this point must not exceed a certain limit value. The vector of general stress $\vec{f}$, acting on the elementary area dS with normal vector $\vec{e}_n$, in the surroundings of the point C, characterises the stress values only in this section and bears no information on stresses in the other elementary areas containing the C point but with other orientations. To avoid limit states controlled by stress values, it is necessary that stress values in any of the infinite number of elementary areas containing the point C be lower than a certain stress limit. Only a complete set of general stresses in all these areas describes **the state of stress in the point C**. general stress

State of stress in a point of the body is a set of general stresses in all sections containing this point.

The question is how many sections (elementary areas) and how oriented are necessary for unambiguous determination of the **state of stress** in the point C. It can be documented by mathematical manipulations that components of general stress (i.e. normal and shear stresses) in any elementary area containing the point C can be calculated from general stresses in three perpendicular sections. It is usual to choose such a cartesian coordinate system that its axes are lines of intersection of these three perpendicular planes. The general stresses will be denoted by index corresponding to the normal line of the plane in which the stress acts; e.g. general stress $\vec{f}_x$ acts in the plane with normal line x, i.e. in the coordinate plane yz. Each of the general stresses, the direction of which is inclined to any of the coordinate axes, can be decomposed into components parallel to the axes of the cartesian coordinate system using the following formulas:

$$\begin{aligned}
\vec{f}_x &= \sigma_x \vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k}, \\
\vec{f}_y &= \tau_{yx} \vec{i} + \sigma_y\vec{j} + \tau_{yz}\vec{k}, \\
\vec{f}_z &= \tau_{zx} \vec{i} + \tau_{zy}\vec{j} + \sigma_z\vec{k},
\end{aligned}$$

where $\sigma_i\,(i = x,\,y,\,z)$ are normal stresses, $\tau_{ij}\,(i,\,j = x,\,y,\,z;\,i \neq j)$ are shear stresses; their first subscript i denotes the normal of the plane in which the stress is acting and the second subscript j denotes the direction of the stress (in the case of normal stresses both subscripts are identical and usually only one subscript is used).

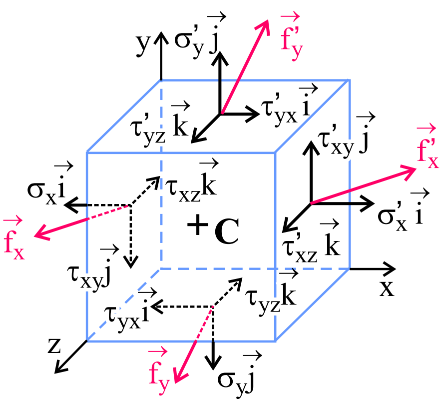
These three general stresses can be organised in a convenient way into a square matrix which represents - in the chosen cartesian coordinate system - the **stress tensor** T_σ:

$$T_\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

Explanation

Stress state in a given point of the body is unambiguously determined by the stress tensor T_σ.

In the linear theory of elasticity, which is based on the assumption of small strains, not all the components of the tensor T_σ are independent. It can be proven using momentum equilibrium equations of the threefold **infinitesimal** element. If we isolate this element as a free body, general stresses $\vec{f_i}$ act in its planar faces (identic with coordinate planes). General stresses $\vec{f_i'}$ act in the opposite faces which are parallel to the coordinate planes.



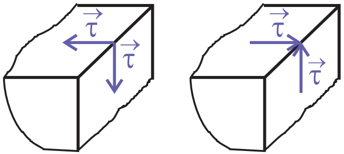
Note to signs of stresses:
In the figure, positive stresses are oriented accordingly to the positive (outer) normals of the sections. It means that positive stresses are oriented in the positive orientation of the corresponding coordinate axes, if the normal of the plane in question is also positively oriented (i.e. in the planes parallel to coordinate planes). In opposite, in the planes with negatively oriented normals (i.e. identical with coordinate planes), the orientation of positive stresses is identic with the negative orientation of the corresponding coordinate axes.

From the momentum equilibrium equations **related** to the point C (identical with the **centroid** of the element) it can be obtained:

$$\sum M_{Cz} = 0 : \left[(\tau_{xy}+\tau'_{xy})\mathrm{d}y\mathrm{d}z\right]\frac{\mathrm{d}x}{2} - \left[(\tau_{yx}+\tau'_{yx})\mathrm{d}x\mathrm{d}z\right]\frac{\mathrm{d}y}{2} = 0 \Rightarrow (\tau_{xy}+\tau'_{xy}) - (\tau_{yx}+\tau'_{yx}) = 0.$$

Because of lucidity, the stresses in the front and rear faces of the element (with normal z) are not drawn in the figure. The resulting force of volumetric forces (e.g. gravity forces) crosses the point C, so that its momentum to this point equals zero. Stresses in the opposite faces of the element are approximately equal, (it reads $\tau_{xy} \to \tau'_{xy}$ and $\tau_{yx} \to \tau'_{yx}$), therefore it reads $\tau_{xy} = \tau_{yx}$. Similarly, momentum equations for components of the momentum $\vec{M}_C$ in directions x and y give formulas $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$.

This result can be interpreted as follows:
The components of the stress tensor T_σ located symmetrically to the principal diagonal of the matrix are identical. In other words, the order of subscripts at shear stresses is not significant.



In general, this result can be rewritten by the following formula: $\tau_{ij} = \tau_{ji}$.
This formula is a mathematical expression for the theorem of shear stress equality and can be formulated as follows:

> Shear stresses on perpendicular faces of an element are equal in magnitude and have directions such that both stresses point toward, or both point away from the line of intersection of the faces.

As a consequence of this theorem, stress state in a point of the body is unambiguously determined by six independent components of the **stress tensor** T_σ, because this tensor can be expressed by a symmetric square matrix.

Stress state in a point of the body is described by the **stress tensor** and depends on the **shape of the body**, its **loads** and on the **position** of the investigated point in the body. In some cases, the stress state can be influenced by **material properties** as well.

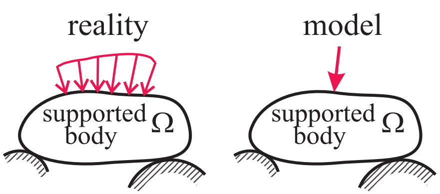
Stress state of the body is a set of stress states in the individual points of the body. It is determined by a **tensor field**, i.e. by a set of stress tensors in all the points of the body. It depends on the **shape of the body** and its **loads**, and, in some cases, it can be influenced by **material properties** as well.

The **stress state of the body** is denoted as **homogeneous** if stress states in all the points of the body are equal, i.e. if stress tensors in all the points of the body are identical.

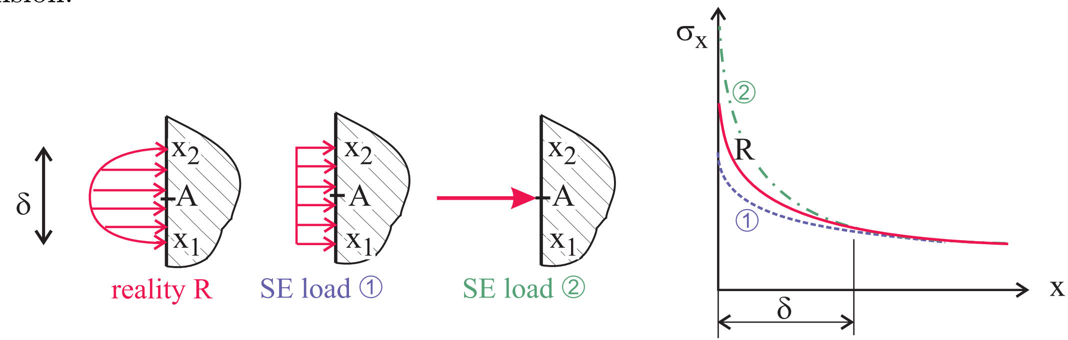
4.1. Saint Venant's principle

When solving practical problems, we usually do not know the real distribution of the area forces acting on the body surface and we must replace them by a model of force interaction with various degree of simplification (isolated force, couple of forces, constant specific force per unit area etc.). This replacement evokes a basic question concerning the applicability of the results in practice: how is the change of the stress state in the body if the system of loads is substituted by another system of loads that acts in the same region of the body surface and is statically equivalent to the original one.

It was proven by comprehensive analyses that difference between the effects (stresses) of two different but statically equivalent loads becomes negligible at a distance at least equal to the largest dimension of the loaded region.



The practical meaning of this principle for stress analyses of bodies and structures can be illustrated in the following figures. If we draw the distribution of one stress component (e.g. σ_x) along a straight line crossing the body, say for the real (reality R) load and for two statically equivalent load substitutes (SE loads 1 and 2), the effect of the load substitution will become insignificant in the distance larger than δ from the loaded region with δ the dimension.

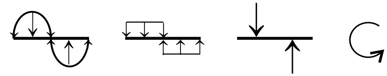


These facts were at first formulated quite intuitively by Saint-Venant; at the actual level of science, **Saint-Venant's principle** can be expressed as follows:

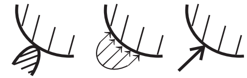
If a real system of loads is substituted by another system of loads, which acts in the same region of the body and is statically equivalent to the original one, the stresses in the body caused by either of the two systems are the same, except of a volume in near surroundings of the loaded region; the dimensions of this volume correspond to the dimensions of the loaded region.

Importance of Saint-Venant's principle:

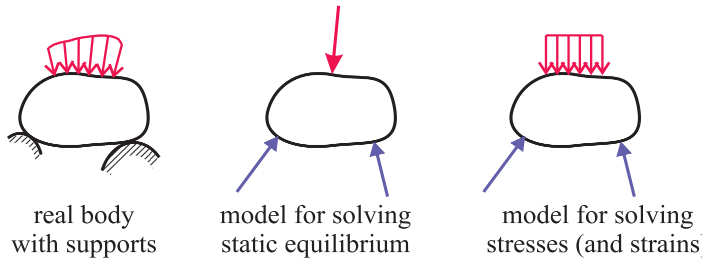
a) it enables us to use computational models of loads (volume and area forces) correctly



b) it enables us to introduce computational models of contact between bodies correctly

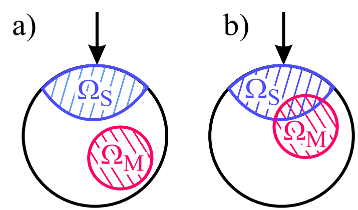


c) it proves incorrectness of some substitutes (commonly used in statics) for stress analyses



Any substitution in stress analysis should be, in addition to the static equivalence, evaluated in accordance with the Saint-Venant's principle. The acceptability of the substitution, however, depends on the limit states significant for the body in question.

In general, the substitution of a system of loads by another one is always acceptable, if the region Ω_S where the substitution of loads was carried out, is quite different from the region Ω_M where limit states are expected. If these two regions have some common points the substitution is acceptable only in some special cases as described below.



If the regions Ω_S and Ω_M has some common part, the substitution is acceptable under the following conditions:

- The region Ω_S where the substitution is carried out is relatively small in comparison with the body.
- The risk of failure is rather higher in the body loaded by the substitutive system of loads than under the real loads.