Continuum mechanics

Lecture 5

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Boundary-Value Problems of Mechanics

We summarize the equations mentioned above

- Equation of motion

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} \quad \text{or} \quad \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ji}}{\partial x_j} + f_i.$$

- Strain-displacement equations

$$\boldsymbol{e} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla\right) \quad \text{or} \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

- Linear constitutive relations

$$\boldsymbol{\sigma} = \boldsymbol{c} : \boldsymbol{e} - \boldsymbol{\beta}(T - T_0) \quad \text{or} \quad \sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij}(T - T_0).$$

Boundary-Value Problems of Mechanics

Thus, there are 15 unknowns and 15 equations for a three-dimensional elastic body, which should be solved in conjunction with appropriate initial and boundary conditions.

Variables	Number
$\boldsymbol{u}$, u_i	3
$\boldsymbol{e}$, e_{ij}	6
$\boldsymbol{\sigma}$, σ_{ij}	6

Boundary-Value Problems of Mechanics

For a three-dimensional elastic body, the initial and boundary conditions are of the following form

- Initial conditions ($t = 0$, $\boldsymbol{x}$ or x_i inside the body)

$$\boldsymbol{u} = \boldsymbol{u}^0, \ \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{v}^0 \quad \text{or} \quad u_i = u_i^0, \ \frac{\partial u_i}{\partial t} = v_i^0.$$

- Boundary conditions ($t \geq 0$, $\boldsymbol{x}$ or x_i on the boundary)
  - geometric or essential type:

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{or} \quad u_i = \hat{u}_i,$$

  - forced or natural type:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \hat{\boldsymbol{t}} \quad \text{or} \quad \sigma_{ij} n_j = \hat{t}_i.$$

Here, variables with a superscript or subscript zero denote the specified initial values and those with a caret over them denote the specified boundary values of the variables.

Boundary-Value Problems of Mechanics

Solving the 15 equations for the 15 unknowns in conjunction with the initial and boundary conditions is of primary interest in solid mechanics.

Boundary-Value Problems of Mechanics

We can classify the formulations as follows (for isothermal case):

1. *Displ. Formulation (Boundary-Value Problem of 1st Kind).* The determination of displacements and stresses in the interior of the body under a given body force distribution and specified displacement field over the entire boundary of the body.
2. *Stress Formulation (Boundary-Value Problem of 2nd Kind).* The determination of displacements and stresses in the interior of the body under the action of a given body force distribution and specified surface forces on the entire boundary.
3. *Mixed Formulation (Boundary-Value Problem of 3rd Kind).* The determination of displacements and stresses in the interior of the body under the action of a given body force distribution and specified displacement field on a portion of the boundary and specified boundary forces on the remaining portion of the boundary of the body.

Boundary-Value Problems of Mechanics

For the boundary-value problem of 1st kind, it is convenient to express all of the governing equations in terms of the displacement field u_i by eliminating stresses σ_{ij} in the equations of motion. The initial conditions fora whole body are

$$u_i = u_i^0, \ \frac{\partial u_i}{\partial t} = v_i^0$$

and boundary conditions are

$$u_i = \hat{u}_i.$$

For an isotropic body with $T = T_0$, the governing equations have form

$$(\lambda + \mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

These equations are called the *Navier equations* of motion.

Boundary-Value Problems of Mechanics

For the boundary-value problems of 2[nd] kind, it is convenient to write the governing equations in terms of stresses σ_{ij}. It is the simplest boundary value problem in the case of an isothermal loading process, suitable for finding a solution in analytical form. The equations of motion are reduced to the form

$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0$$

with prescribed natural boundary conditions

$$\sigma_{ij} n_j = \hat{t}_i.$$

Boundary-Value Problems of Mechanics

The extreme values of stress σ_{ij} are usually of interest in solved elasticity problems, but if the goal is also to know the deformations of the body e_{ij}, it is necessary to solve additional differential equations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \sigma_{ij} = c_{ijkl}e_{kl} - \beta_{ij}(T - T_0).$$

to obtain the displacements u_i. The displacements u_i can be found uniquely, except for the rigid-body motion.

Boundary-Value Problems of Mechanics

For the boundary-value problem of 3rd kind, the governing equations are given by (for $t > 0$)

$$\frac{\partial \sigma_{ki}}{\partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad \sigma_{ij} = \frac{1}{2} c_{ijkl}(u_{k,l} + u_{l,k})$$

with boundary conditions

$$u_i = \hat{u}_i,$$

on one portion and

$$\sigma_{ij} n_j = \hat{t}_i,$$

on the remainder of the boundary. Most problems of solid mechanics fall under this 3rd (i.e. mixed) category.

Existence and Uniqueness of Solutions

The existence and uniqueness of the solution of the above mentioned differential equations follows from *the Lax-Milgram theorem*, which assumes continuity and positive-definiteness of the operator in the governing differential equation

$$\frac{\partial \sigma_{ki}}{\partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad \sigma_{ij} = \frac{1}{2} c_{ijkl}(u_{k,l} + u_{l,k}),$$

which in the case of its linearity and linearity of stress-strain relations is satisfied. Suppose that u_i^1 and u_i^2 are two solutions of the same mixed boundary-value problem.

Existence and Uniqueness of Solutions

We assume the following initial conditions and mixed boundary conditions

$$u_i = u_i^0, \ \frac{\partial u_i}{\partial t} = 0 \quad \text{at } t = t_0$$

$$u_i = \hat{u}_i \quad \text{on } S_1, \quad \sigma_{ji} n_j = \hat{t}_i \quad \text{on } S_2.$$

Existence and Uniqueness of Solutions

Then we have

$$\frac{\partial \overline{\sigma}_{ji}}{\partial x_j} - \rho \frac{\partial^2 \overline{u}_i}{\partial t^2} = 0,$$

where

$$\overline{\sigma}_{ij} = \sigma_{ij}^1 - \sigma_{ij}^2, \quad \overline{u}_i = u_i^1 - u_i^2.$$

Multiply this equations by $\partial \overline{u}_i / \partial t$ and integrate first over the volume of the body and then with respect to time from initial time t_0 to current time t

$$\int_{t_0}^{t} \int_{\Omega} \left[\frac{\partial}{\partial x_j} \left(\frac{\partial \overline{U}_0}{\partial \overline{e}_{ij}} \right) - \rho \frac{\partial^2 \overline{u}_i}{\partial t^2} \right] \frac{\partial \overline{u}_i}{\partial t} d\Omega dt = 0.$$

Existence and Uniqueness of Solutions

Consequently integrating previous integral per-partes, it holds

$$\int_{t_0}^{t}\int_{\Omega}\left(-\frac{\partial \overline{U}_0}{\partial \overline{e}_{ij}}\frac{\partial^2 \overline{u}_i}{\partial x_j \partial t} - \rho\frac{\partial^2 \overline{u}_i}{\partial t^2}\frac{\partial \overline{u}_i}{\partial t}\right) d\Omega dt + \int_{t_0}^{t}\int_{S} n_j \frac{\partial \overline{U}_0}{\partial \overline{e}_{ij}}\frac{\partial \overline{u}_i}{\partial t} dS dt = 0.$$

The surface integral vanishes, because

$$\frac{\partial \overline{u}_i}{\partial t} = 0 \quad \text{on } S_1 \quad \text{and} \quad \frac{\partial \overline{U}_0}{\partial \overline{e}_{ij}} n_j = 0 \quad \text{on } S_2.$$

Moreover, it holds

$$\frac{\overline{U}_0}{\partial \overline{e}_{ij}}\frac{\partial^2 \overline{u}_i}{\partial x_j \partial t} = \frac{\overline{U}_0}{\partial \overline{e}_{ij}}\frac{\partial \overline{e}_{ij}}{\partial t} = \frac{\partial \overline{U}_0}{\partial t}.$$

Existence and Uniqueness of Solutions

Then we have

$$-\int_\Omega \left(\overline{U}_0 + \frac{1}{2}\rho \frac{\partial \overline{u}_i}{\partial t}\frac{\partial \overline{u}_i}{\partial t}\right) d\Omega\bigg|_t + \int_\Omega \left(\overline{U}_0 + \frac{1}{2}\rho \frac{\partial \overline{u}_i}{\partial t}\frac{\partial \overline{u}_i}{\partial t}\right) d\Omega\bigg|_{t_0} = 0$$

The second integral we set zero, because both strain and kinetic energy U_0 and K, respectively, are measured from initial values $U_0(t_0) = \partial u(t_0)/\partial t = 0$. Thus we have

$$\int_\Omega \left(\overline{U}_0 + \frac{1}{2}\rho \frac{\partial \overline{u}_i}{\partial t}\frac{\partial \overline{u}_i}{\partial t}\right) d\Omega = 0.$$

Existence and Uniqueness of Solutions

Both expressions

$$\overline{U}_0 = \frac{1}{2} c_{ijkl} \overline{e}_{ij} \overline{e}_{kl} \quad \text{and} \quad \frac{1}{2} \rho \frac{\partial \overline{u}_i}{\partial t} \frac{\partial \overline{u}_i}{\partial t}$$

are quadratic functions of $\overline{e}_{ij}$ and $\partial \overline{u}_i / \partial t$ in

$$\int_\Omega \left(\overline{U}_0 + \frac{1}{2} \rho \frac{\partial \overline{u}_i}{\partial t} \frac{\partial \overline{u}_i}{\partial t} \right) d\Omega = 0.$$

from which follows that

$$\overline{U}_0 = 0 \quad \text{and} \quad \frac{\partial \overline{u}_i}{\partial t} = 0 \quad \text{everywhere in } \Omega.$$

Existence and Uniqueness of Solutions

These in turn require that

$$u_i^1 = u_i^2 + k,$$

where k is an arbitrary constant. In other words, the displacements of the mixed boundary-value problem are unique within an arbitrary constant responsible for the rigid-body motion. However this constant must be equal zero, because both u_i^1 and u_i^2 are equal to $\hat{u}_i$ on S_1. From this follows the uniqueness of solutions. When u_i is not specified at any point of the boundary (like in the boundary-value problem of 2$^{\text{nd}}$ kind), the stresses σ_{ij} are unique but the displacements u_i are not and they differ by a rigid-body motion.

Axially Loaded Bars

The equations of axially loaded members are based on the assumption of an uniform stress distribution σ_x as a function of x alone and is independent of the position at any given section.

Axially Loaded Bars

The main restrictions of the theory of axially loaded members called bars are

1. The cross section of the member is arbitrary, but is either uniform or gradually varying in the axial direction. If sudden changes in the cross section are involved, the centroids of all cross sections can be joined by a straight line parallel to the axis of the member.
2. The material of the member is homogeneous or the modulus of elasticity E can be a function of the axial coordinate.
3. All applied loads and support points are geometrically positioned in line with the centroidal axis of the member.
4. The load magnitude in compression is less than the critical buckling load of the member.
5. The points of load application and support connections are at reasonable distances from the point of interest.

Axially Loaded Bars

The previous restriction allows one to formulate the mixed boundary-value problem to the form

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + f = 0, \quad 0 < x < L, \quad \sigma_x = E\frac{du}{dx},$$

where L is the length of the bar, A is the cross-sectional area and E is the Young's modulus. The one type of boundary conditions can be prescribed at the boundary points. They can be expressed in terms of the displacements as

$$u = \hat{u}, \quad AE\frac{du}{dx} = \hat{P},$$

where $\hat{P}$ is the prescribed external axial load.

Theory of straight beams

Beams are probably the most common type of structural members. In linear analyses, the individual effects of bending, twisting, buckling and axial deformation can be superimposed. To study the bending effects alone, we place additional restrictions to the above discussed restrictions in the case of axial loaded bars.

Theory of straight beams

These additional restrictions on the geometry and loading of beams are as follows

1. The cross section of the beam has a longitudinal plane of symmetry.
2. The resultant of the transversely applied loads lies in the longitudinal plane of symmetry.
3. Plane sections originally perpendicular to the longitudinal axis of the beam *remain plane and perpendicular* to the longitudinal axis after bending.
4. In the deformed beam, the planes of cross sections have a common intersection, that is, any line originally parallel to the longitudinal axis of the beam becomes an arc of a circle.

Theory of straight beams

These restrictions with restriction on axial loaded beam form following mixed boundary-value problem

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + f = 0, \quad 0 < x < L,$$

$$\sigma_x = -Ez\frac{d^2w}{dx^2} = \frac{M_y}{I}z, \quad \sigma_{zx} = E\frac{d^3w}{dx^3}\frac{Q_y}{h} = \frac{dM_y}{dx}\frac{Q_y}{Ih} = \frac{V_z Q_y}{Ih},$$

where L denotes the length of the beam, h is the width of the cross section, E is the Young's modulus and I is the moment of inertia of the cross section about the y-axis perpendicular to the plane of beam and its external loadings, i.e. zx-plane.

Theory of straight beams

Internal moment M_y and shear force V_z are the resultants of the elementary moments of normal forces and tangential forces across the beam cross section

$$M_y = -\int_A \sigma_x z dA, \quad V_z = \int_A \sigma_{zx} dA$$

and must be in equilibrium with external volume forces

$$\frac{dM_y}{dx} = V_z, \quad \frac{d^2 M_y}{dx^2} = -f.$$

The moment Q_y of the area A' bounded by line z and the top surface of the beam about the neutral axis is

$$Q_y = \int_{A'} z dA.$$

Theory of straight beams

The boundary condition specified at the ends of the beam can be

$$w = \hat{w} \text{ or } V_z = \hat{V} \quad \text{and} \quad \frac{dw}{dx} = \hat{\theta} \text{ or } M_y = \hat{M},$$

where $\hat{w}$, $\hat{\theta}$, $\hat{V}$ and $\hat{M}$ are prescribed external deflection, rotation, shear force and moment, respectively.

Torsion of prismatic member

The main restrictions of the torsion theory of prismatic bars are

1. The material is homogeneous and obeys Hook's law.
2. The cross section of the member is arbitrary but constant along the length.
3. The warping of cross section is permitted, but is taken to be independent of axial location.
4. The projection of any warped cross section rotates as a rigid body, and the angle of twist per unit length is constant.

Torsion of prismatic member

These restrictions allows one to take the form of equilibrium equation as follows

$$-\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)=2G\theta \quad \text{for } (x,y)\in A,$$

$$\sigma_x=\sigma_y=\sigma_z=\sigma_{xy}=0,$$

$$\sigma_{xz}=G\left(\frac{\partial w}{\partial x}-\theta y\right), \quad \sigma_{yz}=G\left(\frac{\partial w}{\partial y}+\theta x\right),$$

where A is the domain of the cross section, G is the shear modulus and θ is the angle of twist and w is the warping function.

Torsion of prismatic member

The function $\Phi(x, y)$ is known as the *Prandtl stress function* and it is defined as

$$\sigma_{xz} = \frac{\partial \Phi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial \Phi}{\partial x}$$

and must obeying the boundary condition

$$\Phi = 0 \quad \text{on } S,$$

where S is the boundary of the cross section. The applied twisting moment M_z is related to the Prandtl stress function as

$$M_z = \int_A (\sigma_{yz} z - \sigma_{xz} y) dA = 2 \int_A \Phi dx dy.$$

Plane elasticity

Consider a linear elastic solid of uniform thickness h bounded by two parallel planes $z = -h/2$ and $z = h/2$ and by any closed boundary S. When the thickness :math:'h' is very large, the problem is considered to be a *plain-strain problem*, and when the thickness is small compared with the lateral dimension (x, y in this case), the problem is considered to be a *plane-stress problem*. Both of these problems are simplifications of three-dimensional elasticity problems under the following assumptions on loading

1. The body forces and applied boundary forces, if any, do not have components in the thickness direction.
2. The body forces do not depend on the thickness coordinate, and the applied boundary forces are uniformly distributed across the thickness.
3. No loads are applied on the parallel planes bounding the top and bottom surfaces.

Plane elasticity

The assumption that the force are zero on the parallel planes implies that the stresses associated with the z-direction are negligibly small for plane stress problems

$$\sigma_z = \sigma_{yz} = \sigma_{xz} = 0.$$

For plane-strain problems, the assumption is that the strains associated with the z-direction are zero (in generalized plane-strain problems, they are assumed to be constant)

$$e_z = e_{xz} = e_{yz} = 0.$$

Plane elasticity

An example that illustrates the difference between plane-stress and plane-strain problems is provided by the bending of a rectangular cross-section beam. If the beam is narrow, it is considered a plane-stress problem. If the beam is very wide, it is considered a plane-strain problem.

Plane elasticity

The equations governing the two types of plane elasticity problem discussed above can be obtained from the three-dimensional equations by using the assumptions of zero stresses or strains in the z-direction

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0,$$

with the constitutive equations

$$\sigma_x = c_{11} e_x + c_{12} e_y,$$
$$\sigma_y = c_{12} e_x + c_{22} e_y,$$
$$\sigma_{xy} = 2 c_{66} e_{xy},$$

for $(x, y) \in \Omega$, Ω is the plane domain and f_x and f_y are coordinates of body forces.

Plane elasticity

The strain-displacement relations are reduced to

$$e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad 2e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

Plane elasticity

The elasticity constants for orthotropic material are given as follows

$$c_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad c_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}},$$

$$c_{12} = \nu_{12}c_{22} = \nu_{21}c_{11}, \quad c_{66} = G_{12} \qquad \text{for plane stress}$$

or

$$c_{11} = \frac{E_1(1 - \nu_{12})}{(1 + \nu_{12})(1 - \nu_{12} - \nu_{21})}, c_{22} = \frac{E_2(1 - \nu_{21})}{(1 + \nu_{21})(1 - \nu_{12} - \nu_{21})},$$

$$c_{12} = \nu_{12}c_{22} = \nu_{21}c_{11}, \quad c_{66} = G_{12} \qquad \text{for plane strain,}$$

where E_1 and E_2 are Young's moduli in one and two material directions, ν_{ij} is Poisson's ratio for transverse strain in the j-th direction when stressed in the i-th direction, and G_{12} is the shear moduli in the 1-2 plane.

Plane elasticity

For isotropic material we have

$$c_{11} = c_{22} = \frac{E}{1-\nu^2}, \quad c_{12} = \frac{\nu E}{1-\nu^2},$$

$$c_{66} = \frac{E}{2(1+\nu)} \quad \text{for plain stress,}$$

or

$$c_{11} = c_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, c_{12} = \frac{E\nu(1-\nu)}{(1+\nu)(1-2\nu)},$$

$$c_{66} = \frac{E}{2(1+\nu)} \qquad \text{for plain strain.}$$

Plane elasticity

The natural and essential boundary conditions are as follows

$$\sigma_x n_x + \sigma_{xy} n_y = \hat{t}_x,$$
$$\sigma_{xy} n_x + \sigma_y n_y = \hat{t}_y \quad \text{on } S_2$$

and

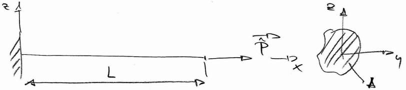
$$u = \hat{u}, \quad v = \hat{v} \quad \text{on } S_1,$$

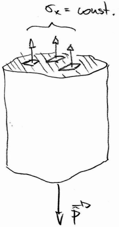
where $\boldsymbol{n} = (n_x, n_y)$ denotes unit normal to the boundary S, S_1 and S_2 are disjunct portions of the boundary S, $\hat{\boldsymbol{t}} = (\hat{t}_x, \hat{t}_y)$ denotes specified boundary force and $\hat{\boldsymbol{u}} = (\hat{u}, \hat{v})$ denotes specified boundary displacement.

Boundary-Value Problems of Mechanics: Examples

Example 6:

Consider an axially loaded member of arbitrary, but constant cross section and let P be the resultant axial force parallel to the x-axis. Derive the expressions for displacements and stresses inside the member. Assume the necessary restrictions for bars.





The equilibrium equation (for $\sigma_y = \sigma_z = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$) with assumption of zero volume forces is

$$\frac{d}{dx}\sigma_x = 0$$

The strain-displacement and constitutive equations are

$$e_x = \frac{du}{dx}, \quad \sigma_x = E e_x \qquad (\sigma_y = \sigma_z = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0)$$

From these follows

1. $\frac{d}{dx}\sigma_x = 0 \Rightarrow \sigma_x = \text{const} = \frac{P}{A}$

2. $\frac{d}{dx}(E e_x) = 0$

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) = 0$$

Boundary-Value Problems of Mechanics: Examples

This equation can be generalized for varying cross-sectional area

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) = 0$$

From the boundaries conditions and for homogeneous material we get

$$\left.\begin{array}{l} x = 0 \Rightarrow u = 0 \\ x = L \Rightarrow E\frac{du}{dx} = \frac{\hat{P}}{A} \end{array}\right\} \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = C \Rightarrow u = Cx + D$$

$$\Rightarrow x = 0 : \ 0 = C \cdot 0 + D \Rightarrow D = 0$$

$$\Rightarrow x = L : \ \frac{\hat{P}}{A} = CE \Rightarrow C = \frac{\hat{P}}{AE}$$

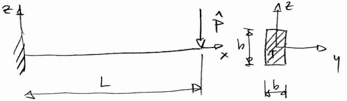
The solution is: $u = \frac{\hat{P}}{AE}x$

$$\frac{du}{dx} = e_x = \frac{\hat{P}}{AE} \Rightarrow \sigma_x = \frac{\hat{P}}{A} \Rightarrow P = \hat{P}$$

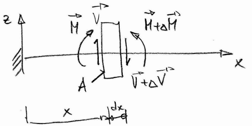
Boundary-Value Problems of Mechanics: Examples

Example 4:

Consider transverse loaded beam by the force $\hat{P}$. Assume the necessary restrictions for beams and derive the expressions for normal stress, displacements inside the beam.



The beam element:



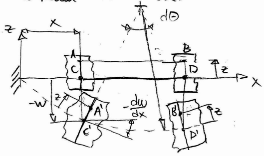
From the equilibrium of the beam element follows

$$V - (V + \Delta V) = 0 \Rightarrow \frac{\Delta V}{\Delta x} = 0 \xrightarrow{\Delta x \to dx} \frac{dV}{dx} = 0$$

$$M + \Delta M - M - V\Delta x = 0 \Rightarrow \frac{\Delta M}{\Delta x} = V \xrightarrow{\Delta x \to dx} \frac{dM}{dx} = V$$

$$\Rightarrow \frac{d^2 M}{dx^2} = 0$$

The beam deformation:



$CD \approx C'D' \approx AB$

$$e_x = \frac{A'B' - AB}{CD} \approx \frac{A'B' - AB}{AB} = \frac{(\varkappa - z)\,d\Theta}{\varkappa\, d\Theta} - 1 = -\frac{z}{\varkappa}$$

From differential geometry follows

$$\frac{1}{\varkappa} = \frac{w''}{(1 + w'^2)^{\frac{3}{2}}} \approx \frac{d^2 w}{dx^2} \quad \text{for } w \ll 1 \text{ and } w' = \frac{dw}{dx}$$

Boundary-Value Problems of Mechanics: Examples

From this we get

$$e_x = -\frac{d^2w}{dx^2} z \quad \text{and assume } e_y \approx 0,\ e_z \approx 0$$

$$\Rightarrow u = -\frac{dw}{dx} z,\quad v = 0,\quad w = w(x)$$

$$\Rightarrow \sigma_x = E e_x = -E \frac{d^2w}{dx^2} z,\quad \sigma_y = \sigma_z = 0$$

$$\Rightarrow M = \int_A \sigma_x z\, dA = \int_A (-E) \frac{d^2w}{dx^2} z^2\, dA = -E \frac{d^2w}{dx^2} \int_A z^2\, dA = -EI \frac{d^2w}{dx^2}$$

So

$$-EI \frac{d^2w}{dx^2} = \hat{P} \cdot (L - x)$$

$$-EI \frac{dw}{dx} = \hat{P} \left(Lx - \frac{x^2}{2}\right) + C$$

$$-EI w = \hat{P} \left(L \frac{x^2}{2} - \frac{x^3}{6}\right) + Cx + D$$

From the boundary conditions we get displacements

$$x = 0:\ w = 0 \wedge \frac{dw}{dx} = 0 \Rightarrow \left.\begin{matrix} 0 = D \\ 0 = C \end{matrix}\right\} \Rightarrow w = -\frac{\hat{P}}{2EI}\left(L - \frac{1}{3}x\right)x^2$$

$$u = \frac{\hat{P}}{2EI}\left(2Lx - x^2\right) z$$

Boundary-Value Problems of Mechanics: Examples

and stress

$$\sigma_x = +E\frac{\hat{P}}{2EI}(2L-2x)z = \frac{\hat{P}}{I}(L-x)z$$

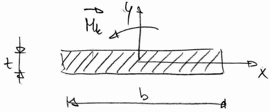
Here

$$I = \frac{1}{12}bh^3$$

Boundary-Value Problems of Mechanics: Examples

Example 8

Consider a thin-walled cross section under the torsion. The dimensions of the cross-section are $t \times b$, where $t \ll b$. The twisting moment is M_k. Find the stresses and displacements over the cross-section.



The Prandtl stress function

$$-\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 2G\Theta$$

is reduced to

$$-\frac{d^2\phi}{dy^2} = 2G\Theta ,$$

because ϕ is essentially constant in the x-direction

$$\phi = -2G\Theta y^2 + C_1 y + C_2$$

From the boundary conditions follows

$$y = \pm\frac{t}{2} : \phi = 0 \Rightarrow \left.\begin{array}{l} C_1 = 0 \\ C_2 = \dfrac{G\Theta t^2}{4} \end{array}\right\} \Rightarrow \phi = -G\Theta\left(y^2 - \frac{t^2}{4}\right)$$

We apply the relation between the twisting moment and Prandtl stress function

$$M_k = 2\int_A \phi \, dx\, dy = 2\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{t}{2}}^{\frac{t}{2}} (-G)\Theta\left(y^2 - \frac{t^2}{4}\right) dy\, dx =$$

Boundary-Value Problems of Mechanics: Examples

$$= -2G\Theta b\left[\frac{y^3}{3} - \frac{t^2}{4}y\right]_{-\frac{t}{2}}^{\frac{t}{2}} = -2G\Theta b\left[\frac{t^3}{24} - \frac{t^2}{4}\frac{t}{2} - \left(-\frac{t^3}{24} + \frac{t^2}{4}\cdot\frac{t}{2}\right)\right] =$$

$$= -2G\Theta b t^3 \frac{1-3+1-3}{24} = +2G\Theta b t^3 \frac{1}{6} = +\frac{G\Theta b t^3}{3}$$

$$\Rightarrow \Theta = \frac{3M_k}{Gbt^3} \Rightarrow \phi = -\not{G}\frac{3M_k}{\not{G}bt^3}\left(y^2 - \frac{t^2}{4}\right) = -\frac{3M_k}{bt^3}\left(y^2 - \frac{t^2}{4}\right)$$

The stress σ_{xz} and σ_{yz} are as follows

$$\sigma_{xz} = \frac{\partial\phi}{\partial y} = -\frac{3M_k}{bt^3}(2y) = -\frac{6M_k}{bt^3}y$$

$$\sigma_{yz} = -\frac{\partial\phi}{\partial x} = 0$$

And finally, the displacement w is

$$\sigma_{xz} = G\left(\frac{\partial w}{\partial x} - \Theta y\right)$$

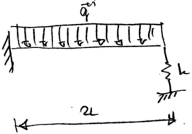
$$\sigma_{yz} = G\left(\frac{\partial w}{\partial y} + \Theta x\right) \Rightarrow 0 = G\left(\frac{\partial w}{\partial y} + \frac{3M_k}{Gbt^3}x\right) \Rightarrow G\frac{\partial w}{\partial y} = -\frac{3M_k}{bt^3}x$$

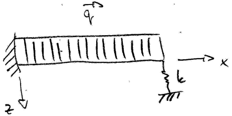
$$w = -\frac{3M_k}{Gbt^3}xy + h(x) \Rightarrow \quad w = -\frac{3M_k}{Gbt^3}xy$$

Setting $w = 0$ for $y = 0$ allows to make zero function $h(x)$ and we get

Boundary-Value Problems of Mechanics: Examples

Example 12: consider a uniform cross-section beam in the figure. Let we find the transverse deflection of the beam.





$$w_{2L} = \frac{6L^4 q}{3EJ + 8L^3 k}$$

We have to solve ordinary diff. equ.

$$EI w^{IV} = q$$

with boundary conditions $w(0) = w'(0) = 0$ and $EI w'''(2L) = k w(2L)$, $EI w''(2L) = 0$

$$EI w^{IV} = q$$

$$EI w''' = qx + c_1$$

$$EI w'' = \frac{1}{2} q x^2 + c_1 x + c_2$$

$$EI w' = \frac{1}{6} q x^3 + \frac{1}{2} c_1 x^2 + c_2 x + c_3$$

$$EI w = \frac{1}{24} q x^4 + \frac{1}{6} c_1 x^3 + \frac{1}{2} c_2 x^2 + c_3 x + c_4$$

Boundary-Value Problems of Mechanics: Examples

From $w(0) = w'(0) = 0$ follows

$$0 = c_3$$

$$0 = c_4$$

From $EI w'''(2L) = k w(2L)$ and $EI w''(2L) = 0$ follows

$$q 2L + c_1 = \frac{k}{EI}\left(\frac{1}{24} q (2L)^4 + \frac{1}{6} c_1 (2L)^3 + \frac{1}{2} c_2 (2L)^2\right)$$

$$\frac{1}{2} q (2L)^2 + c_1 \cdot 2L + c_2 = 0$$

The solution of this algebraic system of equations is

$$c_1 = \frac{-2qL(3EI + 5L^3 k)}{3EI + 8L^3 k}, \quad c_2 = \frac{2L^2 q (3EI + 2L^3 k)}{3EI + 8L^3 k}, \quad c_3 = 0, \quad c_4 = 0$$

For $x = 2L$ we get

$$w(2L) = \frac{6L^4 q}{3EI + 8L^3 k}$$

Boundary-Value Problems of Mechanics: Examples

```
#-----[            ]-----
#-----[ example 12 ]-----
#-----[            ]-----

#-----[ import of sympy library ]-----
import sympy as sp

#-----[ initialization of quality printing ]-----
sp.init_printing()

#-----[ symbol definition ]-----
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')
E,I,q,k,L=sp.symbols('E I q k L')
x=sp.symbols('x')
```

Boundary-Value Problems of Mechanics: Examples

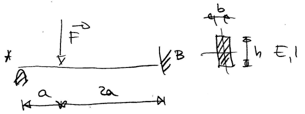
```
#-----[  transevrese deflection of beam   ]-----
#-----[ and its derivaties with respect x ]-----
w=1/E/I*(sp.Rational(1,24)*q*x**4
         +sp.Rational(1,6)*c1*x**3
         +sp.Rational(1,2)*c2*x**2+c3*x+c4)
dw=w.diff(x)
d2w=dw.diff(x)
d3w=d2w.diff(x)

#-----[ boundary conditions of beam ]-----
bc1=w.subs(x,0)
bc2=dw.subs(x,0)
bc3=d2w.subs(x,2*L)
bc4=d3w.subs(x,2*L)-k/E/I*w.subs(x,2*L)
```

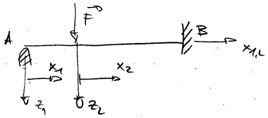
Boundary-Value Problems of Mechanics: Examples

```
#-----[ solution of boundary conditions ]-----
sol=sp.solve([eqn1,eqn2,eqn3,eqn4],[c1,c2,c3,c4])

#-----[ solved transerse deflection of beam ]-----
w_sol=w.subs({c1:sol[c1],c2:sol[c2],
              c3:sol[c3],c4:sol[c4]})
sp.pprint(sp.collect(sp.expand(w_sol),x))

#-----[ transverse deflection in x=2*L ]-----
sp.pprint(sp.simplify(sp.expand(w_sol.subs({x:2*L}))))

#-----[ coefficients c1,c2 ]-----
sp.pprint(sol[c1])
sp.pprint(sol[c2])
```

Boundary-Value Problems of Mechanics: Examples

Example 13: Consider a beam, one end fixed and the second one supported under the external point force F. Derive the transverse deflection.





The equilibrium equations are

$$EIw_1^{IV} = 0, \quad EIw_2^{IV} = 0$$

with boundary and compatibility conditions

$$w_1(0) = 0, \quad w_1''(0) = 0, \quad w_1(a) = w_2(0), \quad w_1'(a) = w_2'(0), \quad w_1''(a) = w_2''(0),$$

$$w_1'''(a) + EIF = w_2'''(0), \quad w_2(2a) = 0, \quad w_2'(2a) = 0.$$

Then we have

$$EIw_1''' = c_1, \quad EIw_1'' = c_1x_1 + c_2, \quad EIw_1' = \frac{1}{2}c_1x_1^2 + c_2x_1 + c_3, \quad EIw_1 = \frac{1}{6}c_1x_1^3 + \frac{1}{2}c_2x_1^2 + c_3x_1 + c_4$$

$$EIw_2''' = d_1, \quad EIw_2'' = d_1x_2 + d_2, \quad EIw_2' = \frac{1}{2}d_1x_2^2 + d_2x_2 + d_3, \quad EIw_2 = \frac{1}{6}d_1x_2^3 + \frac{1}{2}d_2x_2^2 + d_3x_2 + d_4$$

The application of boundary and compatibility equations leads to the system of algebraic equations

$$0 = c_4$$

Boundary-Value Problems of Mechanics: Examples

$$0 = c_2$$

$$\frac{1}{6}c_1 a^3 + \frac{1}{2}c_2 a^2 + c_3 a + c_4 = d_4$$

$$\frac{1}{2}c_1 a^2 + c_2 a + c_3 = d_3$$

$$c_1 a + c_2 = d_2$$

$$c_1 + F = d_1$$

$$0 = \frac{1}{6} d_1 (2a)^3 + \frac{1}{2} d_2 (2a)^2 + d_3 (2a) + d_4$$

$$0 = \frac{1}{2} d_1 (2a)^2 + d_2 2a + d_3$$

from which we get

$$c_1 = -\frac{14}{27}F,\ c_2 = 0,\ c_3 = +\frac{1}{3}Fa^2,\ c_4 = 0$$

$$d_1 = +\frac{13}{27}F,\ d_2 = -\frac{14}{27}Fa,\ d_3 = +\frac{2}{27}Fa^2,\ d_4 = \frac{20}{81}Fa^3$$

and

$$w_1(a) = w_2(0) = w_F = \frac{20Fa^3}{81EI} \approx 0.2469\,\frac{Fa^3}{EI}$$

$$w_1'(a) = w_2'(0) = \varphi_F = \frac{2}{27}\frac{Fa^2}{EI}$$

Boundary-Value Problems of Mechanics: Examples

```
#-----[              ]-----
#-----[ example 13 ]-----
#-----[              ]-----

#-----[ import of sympy library ]-----
import sympy as sp

#-----[ initiation of quality printing ]-----
sp.init_printing()

#-----[ definition of used symbols ]-----
E,I,a,F=sp.symbols('E I a F')
x1,x2=sp.symbols('x1 x2')
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')
d1,d2,d3,d4=sp.symbols('d1 d2 d3 d4')
```

Boundary-Value Problems of Mechanics: Examples

```
#-----[ deflextion of beam, two intervals ]-----
w1=1/E/I*(sp.Rational(1,6)*c1*x1**3
          +sp.Rational(1,2)*c2*x1**2+c3*x1+c4)
w2=1/E/I*(sp.Rational(1,6)*d1*x2**3
          +sp.Rational(1,2)*d2*x2**2+d3*x2+d4)
dw1=w1.diff(x1)
dw2=w2.diff(x2)
d2w1=dw1.diff(x1)
d2w2=dw2.diff(x2)
d3w1=d2w1.diff(x1)
d3w2=d2w2.diff(x2)
```

Boundary-Value Problems of Mechanics: Examples

```
#-----[ boundary and compatibility conditions ]-----
bc1=w1.subs(x1,0)
bc2=d2w1.subs(x1,0)
bc3=w1.subs(x1,a)-w2.subs(x2,0)
bc4=dw1.subs(x1,a)-dw2.subs(x2,0)
bc5=d2w1.subs(x1,a)-d2w2.subs(x2,0)
bc6=d3w1.subs(x1,a)+F/E/I-d3w2.subs(x2,0)
bc7=w2.subs(x2,2*a)
bc8=dw2.subs(x2,2*a)

#-----[ solution of boundary conditions ]-----
sol=sp.solve([bc1,bc2,bc3,bc4,bc5,bc6,bc7,bc8],
             [c1,c2,c3,c4,d1,d2,d3,d4])
```

Boundary-Value Problems of Mechanics: Examples

```
#-----[ result printing ]-----
for ii in [c1,c2,c3,c4]:
  print("\n{}=".format(ii))
  sp.pprint(sol[ii])

for ii in [d1,d2,d3,d4]:
  print("\n{}=".format(ii))
  sp.pprint(sol[ii])

print("\nw1=")
sp.pprint(sp.collect(
           sp.expand(
            w1.subs({c1:sol[c1],c2:sol[c2],
                     c3:sol[c3],c4:sol[c4]})),x1))
```

Boundary-Value Problems of Mechanics: Examples

```
print("\nw2=")
sp.pprint(sp.collect(
           sp.expand(
            w2.subs({d1:sol[d1],d2:sol[d2],
                     d3:sol[d3],d4:sol[d4]})),x2))
```

Thank you!