Continuum mechanics

Lecture 8

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Work and Energy

If $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x})$ is the distributed force acting on a particle occupying position $\boldsymbol{x}$ in the body and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ is the displacement of the particle, then the work done on the particle is $\boldsymbol{F} \cdot \boldsymbol{u}$, and the total work done on the body is the sum of work done on all particles of the body

$$W = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{u} \, d\Omega = \int_\Omega F_i u_i d\Omega,$$

where Ω denotes the volume of the body.

Work and Energy

If a body is subjected to point forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, etc. that displace the points of action by displacements $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, etc., respectively, then the work done by the forces on the body is the sum of the work done by individual forces

$$W = \boldsymbol{F}_1 \cdot \boldsymbol{u}_1 + \boldsymbol{F}_2 \cdot \boldsymbol{u}_2 + \cdots = \sum_i \boldsymbol{F}_i \cdot \boldsymbol{u}_i.$$

Note that in this case, $\boldsymbol{F}_i$ and $\boldsymbol{u}_i$ are not vector functions but constant vectors. Similarly can be obtained for moments $\boldsymbol{M}_i$ and angles $\boldsymbol{\theta}_i$

$$W = \sum_i \boldsymbol{M}_i \cdot \boldsymbol{\theta}_i.$$

Work and Energy

Energy is the capacity to do work. It is a measure of the capacity of all forces that can be associated with matter to perform work. Work is performed on a body through a change in energy. The energy E of a body acted upon by time-dependent forces $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, t)$ is given by the expression

$$E = \int_{t_0}^{t_1} \int_{\Omega} \boldsymbol{F} \cdot \boldsymbol{u} \, d\Omega dt = \int_{t_0}^{t_1} \int_{\Omega} F_i u_i d\Omega dt.$$

Both work and energy are scalars that are independent of the coordinate system used to express them. The choice of the coordinate system only dictates the components of force and displacement, but their product is the same in any coordinate system.

Strain Energy and Complementary Strain Energy

The stress components σ_{ij} can be evaluated from the strain energy density function U_0 under the conditions, that it is independent of the temperature T and the loading process of the body is isothermal, as it was shown in Lecture 4. The existence of a scalar function $U_0 = U_0(e_{ij})$ depending on strains only, from which the stresses are derivable, is of special importance. Such stresses satisfy the energy equation and they are said to be *conservative*. Hence, under the isothermal conditions, we have

$$\frac{\partial U_0}{\partial e_{ij}} = \sigma_{ij} \quad \text{or} \quad dU_0 = \sigma_{ij} de_{ij}.$$

Strain Energy and Complementary Strain Energy

The previous expression is valid for all elastic bodies with linear or nonlinear strain-displacement relations. The complement to the strain energy density U_0 with respect to the double-dot product $\sigma_{ij}e_{ij}$ is the so-called *complementary strain energy density*, U_0^*, which can be computed from

$$U_0^* = \int_0^{\sigma_{ij}} e_{ij} d\sigma_{ij}.$$

For linear elastic materials, we have

$$U_0 = \frac{1}{2} c_{ijkl} e_{ij} e_{kl}, \quad U_0^* = \frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl}$$

and $U_0 = U_0^*$.

Strain Energy and Complementary Strain Energy

The strain energy U and complementary strain energy U^* of an elastic body are given by

$$U = \int_\Omega U_0 d\Omega, \quad U^* = \int_\Omega U_0^* d\Omega.$$

Analogous to the relation between the stresses σ_{ij} and the strain energy density U_0, the strains e_{ij} are computed from

$$e_{ij} = \frac{\partial U_0^*}{\partial \sigma_{ij}}.$$

Strain Energy and Complementary Strain Energy

It should be noted, that both the strain energy density U_0 and the complementary energy density U_0^* can be expressed either in terms of displacements or in terms of forces. The expressions for strain energy U and complementary energy U^* of a beam experiencing bending moment $M_y = M_y(x)$, axial force $N = N(x)$, transverse displacement w and axial displacement u, but neglecting the shear forces $V_z = V_z(x)$, are as follows

$$U = \frac{1}{2}\int_0^L \left[EI \left(\frac{d^2w}{dx^2} \right)^2 + EA \left(\frac{du}{dx} \right)^2 \right] dx,$$

$$U^* = \frac{1}{2}\int_0^L \left(\frac{M_y^2}{EI} + \frac{N^2}{EA} \right) dx.$$

Hamilton's principle

The *work done on the body* at time t by the resultant force in moving through the virtual displacement $\delta \boldsymbol{u}$ is given by the expression

$$\int_{\Omega} \boldsymbol{f} \cdot \delta \boldsymbol{u} \, d\Omega + \int_{S_2} \hat{\boldsymbol{t}} \cdot \delta \boldsymbol{u} \, dS - \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{e} \, d\Omega,$$

where f is the body force vector, $\hat{\boldsymbol{t}}$ is the specified surface stress vector acting on the subdomain S_2 of the boundary S and $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ are the stress and strain tensors.

Hamilton's principle

The symbol δ means variation of the displacements $\delta\boldsymbol{u}$ and strains $\delta\boldsymbol{e}$. It is treated the same as the differential, i.e. for displacements $\boldsymbol{u}$ and $\boldsymbol{v}$ we have

$$\begin{aligned}\delta(\boldsymbol{u} \pm \boldsymbol{v}) &= \delta\boldsymbol{u} \pm \delta\boldsymbol{v}, \\ \delta(\boldsymbol{u}\boldsymbol{v}) &= \delta\boldsymbol{u}\boldsymbol{v} + \boldsymbol{u}\delta\boldsymbol{v}, \\ \delta\left(\frac{\boldsymbol{u}}{\boldsymbol{v}}\right) &= \frac{\delta\boldsymbol{u}\boldsymbol{v} - \boldsymbol{u}\delta\boldsymbol{v}}{\boldsymbol{v}^2},\end{aligned}$$

but unlike the differential, no restrictions to the quality are required on $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ or $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ as the functions of the position $\boldsymbol{x}$.

Hamilton's principle

The variation $\delta \boldsymbol{u}$ or $\delta \boldsymbol{e}$ means the *virtual displacements* or *virtual strains*, respectively, which represent any admissible displacements or deformations of the body at the point $\boldsymbol{x}$ under the conditions that the geometric constrains of the system are not violated and all forces are fixed at their actual equilibrium values.

Hamilton's principle

The term

$$\int_\Omega \boldsymbol{\sigma} : \delta \boldsymbol{e} \, d\Omega$$

in previous expression represents the *virtual work* of internal forces *stored in the body* (as the consequence, it is negative). The strains $\delta \boldsymbol{e}$ are assumed to be compatible in the sense that the strain-displacement relations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

are satisfied.

Hamilton's principle

The work done by the inertia force $m\boldsymbol{a}$ in moving through the virtual displacement $\delta\boldsymbol{u}$ is given by

$$\int_\Omega \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \delta\boldsymbol{u}\, d\Omega,$$

where ρ is the mass density (can be a function of position) of the medium. We have from the Newton's law of motion for continuous body

$$\boldsymbol{F} - m\boldsymbol{a} = 0$$

the following relation

$$\int_{t_1}^{t_2} \left[\left(\int_\Omega \boldsymbol{f} \cdot \delta\boldsymbol{u}\, d\Omega + \int_{S_2} \hat{\boldsymbol{t}} \cdot \delta\boldsymbol{u}\, dS - \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\, d\Omega \right) - \int_\Omega \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \delta\boldsymbol{u}\, d\Omega \right] dt = 0.$$

Hamilton's principle

Integrating the first term of the previous equation by parts

$$\int_{t_1}^{t_2} \int_\Omega \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \cdot \delta \boldsymbol{u}\, d\Omega$$
$$= \left[\rho \frac{\partial \boldsymbol{u}}{\partial t} \cdot \delta \boldsymbol{u}\, d\Omega\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\int_\Omega \rho \frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial \delta \boldsymbol{u}}{\partial t}\, d\Omega\right) dt$$
$$= \left[\rho \frac{\partial \boldsymbol{u}}{\partial t} \cdot \delta \boldsymbol{u}\, d\Omega\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta \left(\int_\Omega \frac{\rho}{2} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial \boldsymbol{u}}{\partial t}\, d\Omega\right) dt,$$

considering $\delta \boldsymbol{u}(t_1) = \delta \boldsymbol{u}(t_2) = 0$, the forces $\boldsymbol{f}$ and $\hat{\boldsymbol{t}}$ to be conservative and that the strain energy density function U_0 exists, then the general form of *Hamilton's principle* for a continuous medium is obtained

$$\delta \int_{t_1}^{t_2} L dt = 0,$$

where L is the *Lagrangian*.

Hamilton's principle

Hamilton's principle for a continuous medium is

$$\delta \int_{t_1}^{t_2} L dt = 0,$$

where the Lagrangian is defined as

$$L = K - (V + U)$$

and K, V and U are the kinetic energy, potential energy of external forces and strain energy, respectively, given by the equations

$$K = \int_{\Omega} \frac{\rho}{2} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, d\Omega,$$

$$V = - \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{u} \, d\Omega - \int_{S_2} \hat{\boldsymbol{t}} \cdot \boldsymbol{u} \, dS,$$

$$U = \int_{\Omega} U_0 d\Omega, \quad U_0 = U_0(e_{ij}), \quad \sigma_{ij} = \frac{\partial U_0}{\partial e_{ij}}.$$

Hamilton's principle

The sum V and U is called the *total potential energy* Π of the body

$$\Pi = V + U$$

and the Lagrangian can be written in the form

$$L = K - \Pi.$$

Hamilton's principle

Integration of the third term in the Lagrangian of Hamilton's principle by parts and applying the divergence theorem, i.e.

$$\int_v \sigma_{ij}\delta e_{ij} d\Omega = \int_\Omega \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j} d\Omega = \int_\Omega \frac{\partial}{\partial x_j}(\sigma_{ij}\delta u_i) d\Omega - \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i d\Omega$$
$$= \int_{S_2} n_j \sigma_{ij}\delta u_i dS - \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i d\Omega$$
$$= \int_{S_2} t_i \delta u_i dS - \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i d\Omega$$

leads to the so-called *Euler-Lagrange equations* associated with the Lagrangian L such that

$$0 = \delta \int_{t_1}^{t_2} L dt$$
$$= \int_{t_1}^{t_2} \left[\int_\Omega \left(\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{div}\boldsymbol{\sigma} - \boldsymbol{f} \right) \cdot \delta \boldsymbol{u} \, d\Omega + \int_{S_2} (\boldsymbol{t} - \hat{\boldsymbol{t}}) \cdot \delta \boldsymbol{u} \, dS \right] dt.$$

Hamilton's principle

$$0 = \int_{t_1}^{t_2} \left[\int_\Omega \left(\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{div}\boldsymbol{\sigma} - \boldsymbol{f} \right) \cdot \delta \boldsymbol{u} \, d\Omega + \int_{S_2} (\boldsymbol{t} - \hat{\boldsymbol{t}}) \cdot \delta \boldsymbol{u} \, dS \right] dt.$$

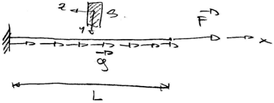
Because $\delta \boldsymbol{u}$ is arbitrary for $t \in (t_1, t_2)$, for $\boldsymbol{x}$ in Ω and also on S_2, it follows that

$$\begin{aligned} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathrm{div}\boldsymbol{\sigma} - \boldsymbol{f} &= 0 \qquad \text{in } \Omega, \\ \boldsymbol{t} - \hat{\boldsymbol{t}} &= 0 \qquad \text{on } S_2. \end{aligned}$$

These are *the Euler-Lagrange equations of elastic body*.

Hamilton's principle: Examples

Example 0.1: Consider a beam loaded as shown in the figure. Derive the essential and natural boundary conditions and determine the Euler equation.



The potential energy of beam is

$$\Pi(u) = \int_0^L \left[\frac{ES}{2}\left(\frac{du}{dx}\right)^2 - g u \right] dx - F \cdot u(L)$$

where u is the axial displacement of a beam
E is the Young's modulus of elasticity

The first variation of the potential $\Pi(u)$ is

$$\delta\Pi(u) = \int_0^L \left[\frac{ES}{2}\, 2\left(\frac{du}{dx}\right)\cdot \frac{d\delta u}{dx} - g\delta u \right] dx - F\cdot\delta u(L),$$

where δu is an arbitrary function of the coordiante x. Integrating by parts we get

$$\delta\Pi(u) = \left\{ \frac{d}{dx}\left(ES\frac{du}{dx}\delta u\right) = \frac{d}{dx}\left(ES\frac{du}{dx}\right)\delta u + ES\frac{du}{dx}\cdot\frac{d}{dx}\delta u \Rightarrow ES\frac{du}{dx}\cdot\frac{d\delta u}{dx} = \frac{d}{dx}\left(ES\frac{du}{dx}\delta u\right) - \frac{d}{dx}\left(ES\frac{du}{dx}\right)\delta u \right\} = \int_0^L \left[\frac{d}{dx}\left(ES\frac{du}{dx}\delta u\right) - \frac{d}{dx}\left(ES\frac{du}{dx}\right)\delta u - g\delta u \right] dx - F\delta u(L)$$

$$= \int_0^L \left[-\frac{d}{dx}\left(ES\frac{du}{dx}\right) - g \right]\delta u\, dx + \left[ES\frac{du}{dx}\delta u \right]_0^L - F\delta u(L)$$

$$= \int_0^L \left[-\frac{d}{dx}\left(ES\frac{du}{dx}\right) - g \right]\delta u\, dx + \left[ES\frac{du}{dx} - F \right]_{x=L}\delta u(L) - ES\frac{du}{dx}\Big|_{x=0}\delta u(0)$$

Hamilton's principle: Examples

Because δu is arbitrary for all $x \in (0, L)$,

the Euler's equation is obtained from the min. of potential energy, i.e. $\delta \Pi(u) = 0$

$$-\frac{d}{dx}\left(ES\frac{du}{dx}\right) - q = 0 \quad \text{pro} \quad 0 < x < L$$

It is assumed, the fulfilling of the boundary condition $\delta u(0) = 0$ for $x = 0$,

then, the last expression in potential is zero automatically and the middle one is zero, if

$$ES\frac{du}{dx} - F = 0 \quad \text{for} \quad x = L,$$

which is the so-called natural boundary condition.

Hamilton's principle: Examples

Example 0.2: Let derive the Euler's equation and the corresponding boundary conditions for the three-dimensional elastic body.

The potential energy of the linearly elastic three-dimnesional body is

$$\Pi(u_i) = \int_V \left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij} - f_i u_i\right) dV - \int_{S_2} \bar{t}_i u_i \, dS ,$$

where the Einstein's summation is used, σ_{ij} is a stress tensor, ε_{ij} is the strain tensor, $\bar{t}_i$ are the tractions and u_i are displacements. S_2 is part of boundary of V, along which the tractions are prescribed and S_1 is the part of the boundary with prescribed u_i like displacements

We suppose the Hooke's law and small deformations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} , \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where λ, μ are Lame constants, δ_{ij} is Kronecker's δ, $u_{i,j}$ is the derivative of ui with respect to xj.

For potential $\Pi(u_i)$ we get

$$\Pi(u_i) = \int_V \left[\frac{1}{2}(2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk})\cdot\varepsilon_{ij} - f_i u_i\right] dV - \int_{S_2} \bar{t}_i u_i \, dS$$

$$\Pi(u_i) = \int_V \left[\frac{1}{2}(2\mu\varepsilon_{ij}\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{ij}\varepsilon_{kk}) - f_i u_i\right] dV - \int_{S_2} \bar{t}_i u_i \, dS$$

$$\Pi(u_i) = \int_V \left[\frac{1}{2}\left(2\mu\cdot\frac{1}{2}(u_{i,j} + u_{j,i})\frac{1}{2}(u_{i,j} + u_{j,i}) + \lambda\delta_{ij}\frac{1}{2}(u_{i,j} + u_{j,i})\cdot\frac{1}{2}(u_{k,k} + u_{k,k})\right) - f_i u_i\right] dV -$$

$$- \int_{S_2} \bar{t}_i u_i \, dS$$

$$\Pi(u_i) = \int_V \left[\frac{1}{4}\mu(u_{i,j} + u_{j,i})^2 + \frac{1}{8}\lambda\cdot(u_{i,i} + u_{i,i})\cdot 2u_{k,k} - f_i u_i\right] dV - \int_{S_2} \bar{t}_i u_i \, dS$$

Hamilton's principle: Examples

$$\Pi(u_i) = \int_V \left[\frac{1}{4}\mu (u_{i,j} + u_{j,i})^2 + \frac{1}{2}\lambda u_{i,i} u_{k,k} - \frac{1}{2} f_i u_i \right] dV - \int_{S_2} \bar{t}_i u_i \, dS$$

The minimum of potential energy $\delta\Pi(u_i) = 0$ gives

$$\delta\Pi(u_i) = \int_V \left[\frac{1}{4}\mu \cdot 2(u_{i,j} + u_{j,i}) \cdot (\delta u_{i,j} + \delta u_{j,i}) + \frac{1}{2}\lambda\, 2 u_{i,i} \delta u_{k,k} - f_i \delta u_i \right] dV - \int_{S_2} \bar{t}_i u_i \, dS = 0$$

$$\delta\Pi(u_i) = \int_V \left[\frac{1}{2}\mu (u_{i,j} + u_{j,i})(\delta u_{i,j} + \delta u_{j,i}) + \lambda u_{i,i} \delta u_{k,k} - f_i \delta u_i \right] dV - \int_{S_2} \bar{t}_i u_i \, dS = 0$$

The integration by parts leads to

$$\int_V \left[\delta u_{i,j} (u_{i,j} + u_{j,i}) + \delta u_i (u_{i,jj} + u_{j,ij}) \right] dV = \int_S \delta u_i (u_{i,j} + u_{j,i}) \cdot n_j \, dS,$$

$$\int_V \left[\delta u_{k,k} u_{i,i} + \delta u_k u_{i,ik} \right] dV = \int_S \delta u_k u_{i,i} n_k \, dS$$

So after the substitution of the expressions

$$\int_V (u_{i,j} + u_{j,i}) \delta u_{i,j} \, dV = -\int_V (u_{i,jj} + u_{j,ij}) \delta u_i \, dV + \int_S (u_{i,j} + u_{j,i}) n_j \delta u_i \, dS$$

$$\int_V u_{i,i} \delta u_{k,k} \, dV = -\int_V u_{i,ik} \delta u_k \, dV + \int_S u_{i,i} n_k \delta u_k \, dS$$

to the condition $\delta\Pi(u_i) = 0$ we get

$$-\int_V \left[\frac{1}{2}\mu \left[(u_{i,jj} + u_{j,ij}) \delta u_i + (u_{i,ji} + u_{j,ii}) \delta u_j \right] + \lambda u_{i,ik} \delta u_k + f_i \delta u_i \right] dV$$

$$+ \int_S \frac{1}{2}\mu \left[(u_{i,j} + u_{j,i}) n_j \delta u_i + (u_{i,j} + u_{j,i}) n_i \delta u_j \right] dS + \int_S \lambda u_{i,i} n_k \delta u_k \, dS - \int_{S_2} \bar{t}_i u_i \, dS = 0$$

Hamilton's principle: Examples

$$-\int_V \left[\mu(u_{i,jj}+u_{j,ij})\delta u_i + \lambda u_{j,ji}\delta u_i + f_i \delta u_i\right] dV + \int_S \left[\mu(u_{i,j}+u_{j,i})n_j\delta u_i + \lambda u_{j,j} n_i \delta u_i\right] dS$$

$$-\int_{S_2} \bar{t}_i \delta u_i \, dS = 0$$

$$-\int_V \left[\mu(u_{i,jj}+u_{j,ij}) + \lambda u_{j,ji} + f_i\right]\delta u_i \, dV + \int_S \left[\mu(u_{i,j}+u_{j,i})n_j\delta u_i + \lambda u_{k,k}\delta_{ij} n_j \delta u_i\right] dS - \int_{S_2} \bar{t}_i \delta u_i \, dS = 0$$

$$-\int_V \left[\mu(u_{i,j}+u_{j,i})_{,j} + \lambda u_{k,ki} + f_i\right]\delta u_i \, dV + \int_S \left[\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij}\right] n_j \delta u_i \, dS - \int_{S_2} \bar{t}_i \delta u_i \, dS = 0$$

$$-\int_V \left[\mu(u_{i,j}+u_{j,i})_{,j} + \lambda u_{k,ki} + f_i\right]\delta u_i \, dV + \int_S \sigma_{ij} n_j \delta u_i \, dS - \int_{S_2} \bar{t}_i \delta u_i \, dS = 0$$

However $\sigma_{ij} n_j = t_i$ and these are non-zero only on part S_2 of boundary S, so

$$-\int_V \left[\mu(u_{i,j}+u_{j,i})_{,j} + \lambda u_{k,ki} + f_i\right]\delta u_i \, dV + \int_{S_2} (t_i - \bar{t}_i)\delta u_i \, dS = 0$$

and we get Euler's equation with natural boundary condition

$$\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,ki} + f_i = 0 \quad \text{ve } V$$

$$t_i = \bar{t}_i \quad \text{na } S_2$$

This eqaution is called Navier's eqaution of the elasticity.

Hamilton's principle: Examples

Ex. 0.3: Let minimalize the functional $I = \frac{1}{2}\int_{t_1}^{t_2}(x\dot{y} - y\dot{x})\,dt$ under the condition that $L = \int_{t_1}^{t_2}(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}dt$, L is aconstant .

The minimization of the functional leads to Euler's equations

$$I_L = \frac{1}{2}\int_{t_1}^{t_2}(x\dot{y} - y\dot{x})dt + \lambda\left(\int_{t_1}^{t_2}(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}dt - L\right) = \int_{t_1}^{t_2}F(x,y,\lambda,\dot{x},\dot{y})\,dt,$$

where λ is the Lagrange's multiplier, which must be constant in this case

$$\delta I_L = 0 = \frac{1}{2}\int_{t_1}^{t_2}(\delta x\dot{y} + x\cdot\delta\dot{y} - \delta y\dot{x} - y\delta\dot{x})dt + \lambda\int_{t_1}^{t_2}\frac{1}{2}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}(2\dot{x}\delta\dot{x} + 2\dot{y}\delta\dot{y})dt$$

$$0 = \frac{1}{2}\int_{t_1}^{t_2}\left[\dot{y}\delta x + \frac{d}{dt}(x\delta y) - \dot{x}\delta y - \dot{x}\delta y - \frac{d}{dt}(y\delta x) + \dot{y}\delta x\right]dt +$$

$$+\lambda\int_{t_1}^{t_2}\left\{\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}(2\dot{x}\delta x + 2\dot{y}\delta y)\right] - \frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}2\dot{x}\right]\delta x - \right.$$

$$\left. - \frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}2\dot{y}\right]\delta y\right\}dt$$

$$0 = \frac{1}{2}\int_{t_1}^{t_2}\left\{\left[2\dot{y} - \frac{d}{dt}\left[\lambda(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}\dot{x}\right]\right]\delta x + \left[-2\dot{x} - \frac{d}{dt}\left[\lambda(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}\dot{y}\right]\right]\delta y\right\}dt$$

$$+\frac{1}{2}\left[x\delta y - y\delta x + \lambda(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}(x\delta x + y\delta y)\right]_{t_1}^{t_2}$$

From the condition that $\delta x = \delta y = 0$ and $(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}(x\delta x + y\delta y) = 0$, we get

Hamilton's principle: Examples

$$0 = \frac{1}{2}\int_{t_1}^{t_2} \left\{ \left[2\dot{y} - \frac{d}{dt}\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \right] \delta x - \left[2\dot{x} + \frac{d}{dt}\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \right] \delta y \right\} dt$$

From which follows

$$\dot{y} - \frac{1}{2}\frac{d}{dt}\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = 0$$

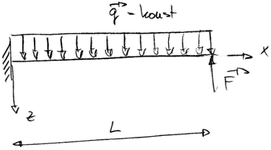
$$\dot{x} + \frac{1}{2}\frac{d}{dt}\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = 0$$

Under the condition that

$$L = \int_{t_1}^{t_2} (\dot{x}^2+\dot{y}^2)^{\frac{1}{2}} dt$$

Hamilton's principle: Examples

Example 0.$: Consider the beam for which we have to find corresponding Euler's equations using the principle of minimum of potential energy.



E - Young's modul of elasticity
J - cross-section moment of inertia.

The potential energy of the beam is

$$\Pi = \int_0^L \left[\frac{EJ}{2} \left(\frac{d\varphi}{dx} \right)^2 + qw \right] dx - F \cdot w(L)$$

under the condition

$$\frac{dw}{dx} + \varphi = 0$$

and the fulfilling of the boundary constions

$$w = \varphi = 0 \quad \text{pro} \quad x = 0,$$

where w is the deflection and φ is the rotation of the beam. We must mimimalize the functional

$$I = \int_0^L \left[\frac{EJ}{2} \left(\frac{d\varphi}{dx} \right)^2 + qw \right] dx - F \cdot w(L) + \int_0^L \lambda \left(\frac{dw}{dx} + \varphi \right) dx$$

Hamilton's principle: Examples

$$0=\delta I=\int_0^L\left[\frac{EJ}{2}2\frac{d\varphi}{dx}\cdot\delta\frac{d\varphi}{dx}+q\delta w\right]dx-F\cdot\delta w(L)+\int_0^L\left[\delta\lambda\left(\frac{dw}{dx}+\varphi\right)+\lambda\left(\delta\frac{dw}{dx}+\delta\varphi\right)\right]dx$$

$$0=\int_0^L\left\{\frac{d}{dx}\left[EJ\frac{d\varphi}{dx}\delta\varphi\right]-\frac{d}{dx}\left[EJ\frac{d\varphi}{dx}\right]\delta\varphi+q\delta w+\left(\frac{dw}{dx}+\varphi\right)\delta\lambda+\frac{d}{dx}\left[\lambda\delta w\right]-\right.$$

$$\left.-\frac{d\lambda}{dx}\delta w+\lambda\delta\varphi\right\}dx-F\delta w(L)$$

$$0=\int_0^L\left\{\left(-\frac{d}{dx}\left[EJ\frac{d\varphi}{dx}\right]+\lambda\right)\delta\varphi+\left(q-\frac{d\lambda}{dx}\right)\delta w+\left(\frac{dw}{dx}+\varphi\right)\delta\lambda\right\}dx-$$

$$-F\delta w(L)+\left[EJ\frac{d\varphi}{dx}\delta\varphi\right]_0^L+\left[\lambda\delta w\right]_0^L$$

Choosing $\delta\varphi=\delta w=0$ for $x=0$ we get

$$0=\int_0^L\left\{\left(-\frac{d}{dx}\left[EJ\frac{d\varphi}{dx}\right]+\lambda\right)\delta\varphi+\left(q-\frac{d\lambda}{dx}\right)\delta w+\left(\frac{dw}{dx}+\varphi\right)\delta\lambda\right\}dx-$$

$$-F\delta w(L)+\left[EJ\frac{d\varphi}{dx}\right]\Big|_{x=L}\delta\varphi(L)+\lambda(L)\delta w(L)$$

From which folows

$$-\frac{d}{dx}\left[EJ\frac{d\varphi}{dx}\right]+\lambda=0 \qquad -F+\lambda=0 \quad \text{pro } x=L$$

$$-\frac{d\lambda}{dx}+q=0 \qquad EJ\frac{d\varphi}{dx}=0 \quad \text{pro } x=L$$

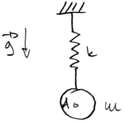
$$\frac{dw}{dx}+\varphi=0$$

and from the equlibrium eqauations we get $\lambda(x)=-T(x)$.

Hamilton's principle: Examples

Example 0.5: Let determine the work of external forces and internal work of spring of the stiffness k.

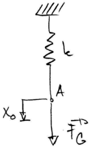
A very slow spring extension is assumed and consequently the dynamic effects can be neglected



$W_I \neq -W_E$

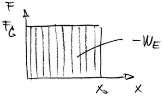
$W_I + W_E \neq 0$ <=

The work of external forces:

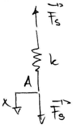


$F_G = m \cdot g$

$W_E = -(F_G \cdot x_0) = -m \cdot g \cdot x_0$

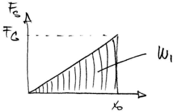


The work of internal forces of the spring



$F_s = k \cdot x$

$W_I = \int_0^{x_0} F_s \, dx = \int_0^{x_0} k \cdot x \, dx = \left[\frac{1}{2} k x^2\right]_0^{x_0} = \frac{1}{2} k x_0^2$

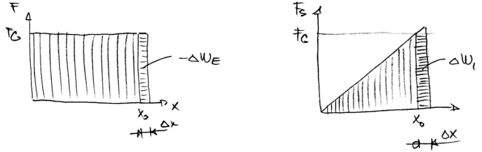


$F_s = F_G$ pro $x = x_0$

Hamilton's principle: Examples

If the static equilibrium is reached $(F_s = F_G)$ let calculate the change of the work of external and internal forces as the consequence of Δx representig the infinitelly small spring extension:

$$\Delta W_E = -F_G \cdot \Delta x = -mg\,\Delta x \qquad \Delta W_I = F_s\,\Delta x = k \cdot x_0 \cdot \Delta x$$

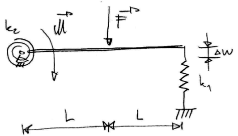


$$\Delta W_I = -\Delta W_E \Rightarrow \Delta W_I + \Delta W_E = 0 \Rightarrow (kx_0 - mg)\,\Delta x = 0$$

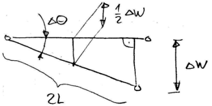
$$\Rightarrow kx_0 - mg = 0 \Rightarrow F_s = F_G$$

Hamilton's principle: Examples

Exaample 0.6: Express the work of the internal load and the internal work of the springs assuming an infinitesimal rotation of the rod from the equilibrium position.



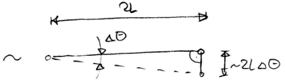
External work:



$$\Delta W_E = -\left(F\,\tfrac{1}{2}\Delta w + \mathcal{M}\cdot\Delta\Theta\right)$$

$$= -\left(F\cdot\tfrac{1}{2}\cdot 2L\sin\Delta\Theta + \mathcal{M}\Delta\Theta\right)$$

$$\sim -\left(F\cdot L\Delta\Theta - \mathcal{M}\Delta\Theta\right) = -\left(F\cdot L + \mathcal{M}\right)\cdot\Delta\Theta$$



Internal work:

$$\Delta W_I = F_S\cdot\Delta w + M_S\cdot\Delta\Theta$$

$$= k_1 w\Delta w + k_2\Theta\Delta\Theta$$

$$\sim \left(k_1 2L\Theta 2L\Delta\Theta + k_2\Theta\Delta\Theta\right)$$

$$= \left(k_1 4L^2 + k_2\right)\Theta\Delta\Theta$$

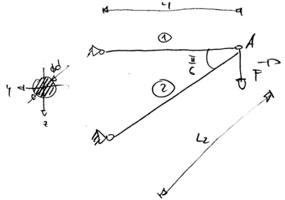
$$\Delta W_I = -\Delta W_E \Rightarrow F\cdot L = k_1 4L^2\Theta,\ \mathcal{M} = k_2\Theta \Rightarrow F = 4k_1 L\Theta,\ \mathcal{M} = k_2\Theta$$

$\Downarrow$

$\Delta W_I + \Delta W_E = 0 \iff$ Virtual displacement principle.

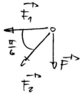
Hamilton's principle: Examples

Example 0.7: Consider the structure in the figure. Express a strain energy density U_0, complementary strain energy density U_0^*, the strain energy U and the complentary strain energy U^*.



$$\sigma = \begin{cases} E\sqrt{\varepsilon} & \text{pro } \varepsilon > 0 \\ -E\sqrt{-\varepsilon} & \text{pro } \varepsilon < 0 \end{cases}$$

1. The statical equilibrium at point A



$$F_1 + F_2 \cdot \cos\frac{\pi}{6} = 0$$
$$F_2 \cdot \sin\frac{\pi}{6} + F = 0$$

$$\frac{1}{2}F_2 + F = 0 \Rightarrow F_2 = -2F$$
$$F_1 - 2F \cdot \frac{\sqrt{3}}{2} = 0 \Rightarrow F_1 = \sqrt{3}F$$

2. The stress in the members of the structure

$$\sigma_1 = \frac{F_1}{S_1} = \frac{\sqrt{3}F}{\frac{\pi d^2}{4}} = \frac{4\cdot\sqrt{3}F}{\pi d^2}$$

$$\sigma_2 = \frac{F_2}{S_2} = \frac{-2F}{\frac{\pi d^2}{4}} = \frac{-8F}{\pi d^2}$$

3. The strain in the members of the structure

$$\varepsilon_1 = \frac{\sigma_1^2}{E^2} = \frac{16\cdot 3\cdot F^2}{\pi^2 d^4 E^2} = \frac{48F^2}{\pi^2 d^4 E^2}$$

$$\varepsilon_2 = -\frac{\sigma_2^2}{E^2} = -\frac{64F^2}{\pi^2 d^4 E^2}$$

4. The strain energy density of the members of the structure

$$U_{01} = \int_0^{\varepsilon_1} \sigma_1 \, d\varepsilon_1 = \int_0^{\varepsilon_1} E\sqrt{\varepsilon_1}\, d\varepsilon_1 = E\left[\frac{2}{3}\varepsilon_1^{\frac{3}{2}}\right]_0^{\varepsilon_1} = \frac{2E}{3}\cdot\varepsilon_1^{\frac{3}{2}} =$$

$$= \frac{2E}{3}\left(\frac{48F^2}{\pi^2 d^4 E^2}\right)^{\frac{3}{2}} = \frac{2E}{3}\left(\frac{4^2\, 3F^2}{\pi^2 d^4 E^2}\right)^{\frac{3}{2}} = \frac{2E}{3}\left(\frac{4\cdot\sqrt{3}F}{\pi d^2 E}\right)^3 =$$

$$= \frac{2E}{3}\cdot\frac{64\cdot 3\sqrt{3}\,F^3}{\pi^3 d^6 E^3} = \frac{128\sqrt{3}F^3}{\pi^3 d^6 E^2}$$

Hamilton's principle: Examples

$$U_{02} = \int_0^{\varepsilon_2} \sigma_2 \, d\varepsilon_2 = \int_0^{\varepsilon_2} -E\sqrt{-\varepsilon_2}\, d\varepsilon_2 = +E\int_0^{-\varepsilon_2'} \sqrt{\varepsilon_2'}\, d\varepsilon_2 = +E\left[\frac{2}{3}(+\varepsilon_2)^{\frac{3}{2}}\right]_0^{-\varepsilon_2} = +\frac{2E}{3}(-\varepsilon_2)^{\frac{3}{2}} =$$

$$= \frac{2E}{3}\left(\frac{8^2F^2}{\pi^2 d^4 E^2}\right)^{\frac{3}{2}} = +\frac{2E}{3}\,\frac{8^3F^3}{\pi^3 d^6 E^3} = \frac{1024\, F^3}{3\pi^3 d^6 E^2}$$

5. The strain energy of the structure

$$U = \int_{V_1} U_{01}\, dV + \int_{V_2} U_{02}\, dV = U_{01} S_1 L_1 + U_{02} S_2 L_2 = \frac{128\sqrt{3}F^3}{\pi^3 d^6 E^2}\,\frac{\pi d^2}{4}\cdot L_1 + \frac{1024 F^3}{3\pi^3 d^6 E^2}\cdot\frac{\pi d^2}{4}\cdot\frac{L_1}{\cos\frac{\pi}{6}} =$$

$$= \frac{F^3L_1}{\pi^2 d^4 E^2}\left(32\sqrt{3} + \frac{256}{3}\cdot\frac{2}{\sqrt{3}}\right) = \frac{32F^3L_1}{\pi^2 d^4 E^2}\left(\sqrt{3} + \frac{8}{3}\,\frac{2\sqrt{3}}{3}\right) = \frac{32\sqrt{3}F^3L_1}{\pi^2 d^4 E^2}\cdot\frac{17}{9} = \frac{544\sqrt{3}F^3L_1}{9\pi^2 d^4 E^2}$$

6. The complementary strain energy density of the members of the structure

$$U_{01}^* = \int_0^{\sigma_1}\varepsilon_1\, d\sigma_1 = \int_0^{\sigma_1}\left(\frac{\sigma_1^2}{E^2}\right)d\sigma_1 = \frac{1}{E^2}\left[\frac{\sigma_1^3}{3}\right]_0^{\sigma_1} = \frac{1}{3E^2}\sigma_1^3 = \frac{1}{3E^2}\left(\frac{4\sqrt{3}F}{\pi d^2}\right)^3$$

$$U_{02}^* = \int_0^{\sigma_2}\varepsilon_2\, d\sigma_2 = \int_0^{\sigma_2}\left(-\frac{\sigma_2^2}{E^2}\right)d\sigma_2 = -\frac{1}{E^2}\left[\frac{\sigma_2^3}{3}\right]_0^{\sigma_2} = -\frac{1}{3E^2}\sigma_2^3 = -\frac{1}{3E^2}\left(-\frac{8F}{\pi d^2}\right)^3 = \frac{1}{3E^2}\left(\frac{8F}{\pi d^2}\right)^3$$

7. The complementary strain energy of the structure

$$U^* = \int_{V_1} U_{01}^*\, dV + \int_{V_2} U_{02}^*\, dV = U_{01}^* S_1 L_1 + U_{02}^* S_2 L_2 = \frac{1}{3E^2}\cdot\frac{4^3\, 3\sqrt{3}\, F^3}{\pi^3 d^6}\cdot\frac{\pi d^2}{4}\cdot L_1 + \frac{1}{3E^2}\,\frac{8^3F^3}{\pi^3 d^6}\cdot\frac{\pi d^2}{4}\cdot\frac{L_1}{\cos\frac{\pi}{6}} =$$

$$= \frac{4^2F^3L_1}{3E^2\pi^2 d^4}\left(3\sqrt{3} + \frac{2^3\cdot 2}{\sqrt{3}}\right) = \frac{16\sqrt{3}F^3L_1}{3E^2\pi^2 d^4}\left(3 + \frac{2^4}{3}\right) = \frac{16\sqrt{3}F^3L_1}{3E^2\pi^2 d^4}\,\frac{25}{3} = \frac{400\sqrt{3}F^3L_1}{9E^2\pi^2 d^4}$$

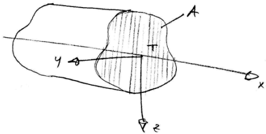
Hamilton's principle: Examples

Example 0.8: Express the strain energy density, the strain energy, the complementary strain energy density and the strain energy density of the beam in classical Euler-Bernoulli beam theory.

The displacements can be written as

$$u = u_0 - z\frac{dw_0}{dx}, \quad v = 0, \quad w = w_0,$$

where (u_0, w_0) are displacements at $(x, 0)$.



$$\varepsilon_{xx} = \frac{du}{dx} = \frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}, \quad \varepsilon_{xy} \sim \varepsilon_{xz} \sim 0$$

$$N = \int_A \sigma_{xx}\, dA, \quad M = \int_A \sigma_{xx} z\, dA, \quad V = \int_A \sigma_{xz}\, dA \quad \Rightarrow \quad -\frac{dN}{dx} = f(x), \quad -\frac{dV}{dx} = q(x), \quad V - \frac{dM}{dx} = 0$$

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right)$$

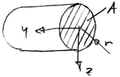
$$N = \int_A E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right) dA = E\frac{du_0}{dx}\int_A dA - E\frac{d^2w_0}{dx^2}\int_A z\, dA = EA\frac{du_0}{dx}$$

$$M = \int_A E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right) z\, dA = E\frac{du_0}{dx}\int_A z\, dA - E\frac{d^2w_0}{dx^2}\int_A z^2 dA = -EI\frac{d^2w_0}{dx^2}$$

$$\Rightarrow \quad \frac{du_0}{dx} = \frac{N}{EA}, \quad \frac{d^2w_0}{dx^2} = -\frac{M}{EI}$$

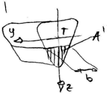
$$A = \int_A dA, \quad I = \int_A z^2 dA$$

$$\sigma_{xx} = \frac{N}{A} + \frac{Mz}{I} \qquad \sigma_{x\theta} = \frac{T r}{J}$$



$$J = \int_A r^2 dA$$

$$\sigma_{xz} = \frac{VQ}{Ib}$$



$$Q = \int_{A'} z^2 dA$$

Hamilton's principle: Examples

The strain energy density:

$$U_0 = \int_0^{\varepsilon_{xx}} \sigma_{xx}\, d\varepsilon_{xx} = \int_0^{\varepsilon_{xx}} E\varepsilon_{xx}\, d\varepsilon_{xx} = \frac{1}{2}E\varepsilon_{xx}^2 = \frac{1}{2}E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right)^2$$

The strain energy:

$$U = \int_V U_0\, dV = \int_0^L \int_A \frac{1}{2}E\left(\frac{du_0}{dx} - z\frac{d^2w_0}{dx^2}\right)^2 dA\, dx = \frac{1}{2}\int_0^L \left[E\left(\frac{du_0}{dx}\right)^2 \int_A dA - 2E\frac{du_0}{dx}\frac{d^2w_0}{dx^2}\int_A z\, dA\right.$$

$$\left. + E\left(\frac{d^2w_0}{dx^2}\right)^2 \int_A z^2 dA\right] dx = \frac{1}{2}\int_0^L \left[EA\left(\frac{du_0}{dx}\right)^2 + EI\left(\frac{d^2w_0}{dx^2}\right)^2\right] dx$$

The strain energy for the torsion:

$$U = \int_V U_0 dV = \int_0^L \int_A \int_0^{\varepsilon_{x\theta}} 2\sigma_{x\theta}\, d\varepsilon_{x\theta}\, dA\, dx = \int_0^L \int_A \int_0^{\varepsilon_{x\theta}} 2\cdot 2\cdot G\,\varepsilon_{x\theta}\, d\varepsilon_{x\theta}\, dA\, dx =$$

$$= \int_0^L \int_A 2G\varepsilon_{x\theta}^2\, dA\, dx = 2\int_0^L GA\, \varepsilon_{x\theta}^2\, dx$$

The complementary strain energy density:

$$U_0^* = \int_0^{\sigma_{xx}} \varepsilon_{xx}\, d\sigma_{xx} + 2\int_0^{\sigma_{xz}} \varepsilon_{xz}\, d\sigma_{xz} + 2\int_0^{\sigma_{x\theta}} \varepsilon_{x\theta}\, d\sigma_{x\theta} = \int_0^{\sigma_{xx}} \frac{1}{E}\sigma_{xx}\, d\sigma_{xx} + 2\int_0^{\sigma_{xz}} \frac{1}{2G}\sigma_{xz}\, d\sigma_{xz}$$

$$+ 2\int_0^{\sigma_{x\theta}} \frac{1}{2G}\sigma_{x\theta}\, d\sigma_{x\theta} = \frac{\sigma_{xx}^2}{2E} + \frac{\sigma_{xz}^2}{2G} + \frac{\sigma_{x\theta}^2}{2G}$$

The complementary strain energy:

$$U^* = \int_V U_0^* dV = \int_0^L \int_A \left(\frac{\sigma_{xx}^2}{2E} + \frac{\sigma_{xz}^2}{2G} + \frac{\sigma_{x\theta}^2}{2G}\right) dA\, dx = \int_0^L \int_A \left[\frac{1}{2E}\left(\frac{N}{A} + \frac{M}{I}z\right)^2 + \frac{1}{2G}\left(\frac{VQ}{Ib}\right)^2\right.$$

$$\left. + \frac{1}{2G}\left(\frac{Tr}{J}\right)^2\right] dA\, dx = \frac{1}{2}\int_0^L \frac{1}{E}\left[\left(\frac{N}{A}\right)^2 \int_A dA + 2\frac{N}{A}\frac{M}{I}\int_A z\, dA + \left(\frac{M}{I}\right)^2 \int_A z^2 dA\right] dx$$

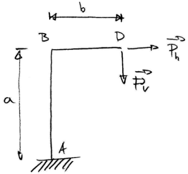
Hamilton's principle: Examples

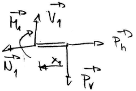
$$+\frac{1}{2}\int_0^L \frac{1}{G}\left(\frac{VQ}{Ib}\right)^2 \int_A dA\,dx + \frac{1}{2}\int_0^L \frac{1}{G}\left(\frac{T}{J}\right)^2 \int_A r^2 dA\,dx = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI}\right) dx + \frac{1}{2}\int_0^L \frac{V^2 f_s}{GA}\,dx$$

$$+\frac{1}{2}\int_0^L \frac{T^2}{GJ}\,dx \quad , \text{where } f_s = \frac{A}{I^2 b^2}\int_A Q^2\,dA$$

Hamilton's principle: Examples

Example 0.9: Express the complementary strain energy density of the structure in the figure.



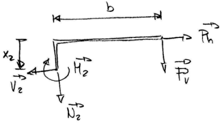


$x_1 \in (0, b)$

$N_1 = P_h$

$V_1 = P_v$

$M_1 = -P_v \cdot x_1$



$x_2 \in (0, a)$

$N_2 = -P_v$

$V_2 = P_h$

$M_2 = -P_v \cdot b - P_h \cdot x_2$

$$U^* = \int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI} \right) dx + \frac{1}{2} \int_0^L \frac{V^2 f_s}{CA} dx$$

$$U^*_{DB} = \int_0^b \left[\frac{P_h^2}{2EA} + \frac{1}{2EI}(-P_v x_1)^2 + \frac{f_s P_v^2}{2GA} \right] dx_1 = \frac{P_h^2 b}{2EA} + \frac{P_v^2 b^3}{6EI} + \frac{f_s P_v^2 b}{2GA}$$

$$U^*_{BA} = \int_0^a \left[\frac{(-P_v)^2}{2EA} + \frac{1}{2EI}(-P_v b - P_h x_2)^2 + \frac{f_s P_h^2}{2GA} \right] dx_2 = \frac{P_v^2 a}{2EA} + \frac{1}{2EI}\Big(P_v^2 b^2 a + 2 \cdot P_v P_h b \frac{1}{2} a^2 +$$

$$+ P_h^2 \frac{1}{3} a^3 \Big) + \frac{f_s P_h^2}{2GA} a = \frac{P_v^2 a}{2EA} + \frac{1}{2EI}\left(P_v^2 a b^2 + P_v P_h a^2 b + \frac{1}{3} P_h^2 a^3 \right) + \frac{f_s P_h^2}{2GA} a$$

Hamilton's principle: Examples

Example 0.11: Calculate the first variation of the following expressions

$$F(u,u',u'') = c_1 u^2 + c_2\left(\frac{du}{dx}\right)^2 + c_3\left(\frac{d^2u}{dx^2}\right)^2 + c_4 u\frac{du}{dx}$$

$$G(u,v,u',v') = c_1 u^2 + c_2 v^2 + c_3 uv + c_4\left(\frac{du}{dx}\right)^2 + c_5\left(\frac{dv}{dx}\right)^2 + c_6\frac{du}{dx}\frac{dv}{dx}$$

Solution:

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' = \left(2c_1 u + c_4\frac{du}{dx}\right)\delta u + \left(2c_2\frac{du}{dx} + c_4 u\right)\frac{d\delta u}{dx} + 2c_3\frac{d^2u}{dx^2}\cdot\frac{d^2\delta u}{dx^2}$$

$$\delta_u G = 2c_1 u\delta u + c_3 v\delta u + 2c_4\frac{du}{dx}\frac{d\delta u}{dx} + c_6\frac{dv}{dx}\frac{d\delta u}{dx} = \left(2c_1 u + c_3 v\right)\delta u + \left(2c_4\frac{du}{dx} + c_6\frac{dv}{dx}\right)\left(\frac{d\delta u}{dx}\right)$$

$$\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' =$$

$$\delta_v G = \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v' = 2c_2 v\delta v + c_3 u\delta v + 2c_5\frac{dv}{dx}\cdot\frac{d\delta v}{dx} + c_6\frac{du}{dx}\frac{d\delta v}{dx} =$$

$$= \left(2c_2 v + c_3 u\right)\delta v + \left(2c_5\frac{dv}{dx} + c_6\frac{du}{dx}\right)\frac{d\delta v}{dx}$$

Hamilton's principle: Examples

Example 0.10: Calculate the first variation of the following functionals

$$I_1 = \int_0^L \left[\frac{a}{2} \left(\frac{du}{dx} \right)^2 + \frac{b}{2} u^2 - fu \right] dx$$

$$I_2 = \int_\Omega \left(\frac{1}{2} \nabla u \cdot \nabla u - fu \right) d\underline{x} - \int_{\Gamma_2} qu \, ds$$

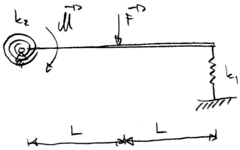
Solution:

$$\delta I_1 = \int_0^L \left[a \frac{du}{dx} \frac{d\delta u}{dx} + b u \delta u - f \delta u \right] dx$$

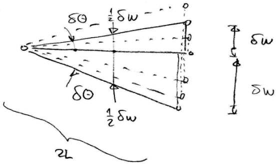
$$\delta I_2 = \int_\Omega \left[\nabla u \cdot \nabla \delta u - f \delta u \right] d\underline{x} - \int_{\Gamma_2} q \delta u \, ds$$

Hamilton's principle: Examples

Example 0.11: Express the virtual work of external forces and internal forces of the rod in the firgure



Solution:



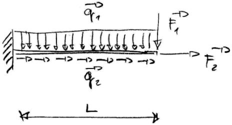
$$\delta W_E = -\left(F\cdot\tfrac{1}{2}\delta w + \mathcal{M}\,\delta\Theta\right) = -\left(F\cdot\tfrac{1}{2}\,2L\sin\delta\Theta + \mathcal{M}\,\delta\Theta\right)$$

$$\sim -\left(F\cdot L\,\delta\Theta + \mathcal{M}\,\delta\Theta\right) = -\left(F\cdot L + \mathcal{M}\right)\delta\Theta$$

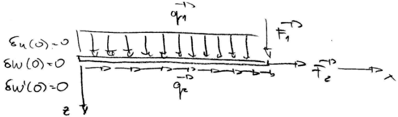
$$\delta W_I = F_s\,\delta w + M_s\,\delta\Theta = k_1 w\,\delta w + k_2\Theta\,\delta\Theta \sim k_1\,2L\Theta\,\delta(2L\Theta) + k_2\Theta\,\delta\Theta = \left(k_1 4L^2 + k_2\right)\Theta\,\delta\Theta.$$

Hamilton's principle: Examples

Example 0.12: For the beam, express the virtual potential energy $\delta\Pi$ and complementary potential energy $\delta\Pi^*$.



Řešení:



$$\delta W_E = -\left[\int_0^L (q_1 \delta w + q_2 \delta u)\,dx + F_1 \delta w(L) + F_2 \delta u(L)\right], \text{ kde } \delta u(0)=0,\ \delta w(0)=0,\ \delta w'(0)=0$$

$$\delta W_i = \int_V \delta U_0\, dV = \int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV = \int_0^L \int_A \sigma_{11}\delta\varepsilon_{11}\, dA\, dx = \int_0^L \int_A \sigma_{xx}\delta\varepsilon_{xx}\, dA\, dx$$

$$\varepsilon_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2} \Rightarrow \delta\varepsilon_{xx} = \frac{d\delta u}{dx} - z\frac{d^2\delta w}{dx^2}, \quad \int_A \sigma_{xx}\, dA = N, \quad \int_A \sigma_{xx}\cdot z\, dA = M$$

$$\delta W_i = \int_0^L \int_A \sigma_{xx}\cdot\frac{d\delta u}{dx}\, dA\, dx - \int_0^L \int_A \sigma_{xx} z \frac{d^2\delta w}{dx^2}\, dA\, dx = \int_0^L \frac{d\delta u}{dx}\int_A \sigma_{xx}\, dA\, dx$$

$$- \int_0^L \frac{d^2\delta w}{dx^2}\int_A \sigma_{xx} z\, dA\, dx = \int_0^L N\frac{d\delta u}{dx}\, dx - \int_0^L M\frac{d^2\delta w}{dx^2}\, dx =$$

$$= \int_0^L \left(N\frac{d\delta u}{dx} - M\frac{d^2\delta w}{dx^2}\right) dx$$

Hamilton's principle: Examples

$$\delta W = \delta W_1 + \delta W_E = \int_0^L \left[N \frac{d\delta u}{dx} - M \frac{d^2 \delta w}{dx^2} - q_1 \delta w - q_2 \delta u \right] dx - F_1 \delta w(L) - F_2 \delta u(L)$$

Let suppose the linearly elastic material $\sigma_{xx} = E \cdot \varepsilon_{xx}$

$$N = \int_A \sigma_{xx}\, dA = \int_A E \varepsilon_{xx}\, dA = E \int_A \frac{du}{dx}\, dA = EA \frac{du}{dx}$$

$$M = \int_A \sigma_{xx}\, z\, dA = \int_A E \varepsilon_{xx} z\, dA = E \int_A (-z) \cdot \frac{d^2 w}{dx^2}\, z\, dA = -E \frac{d^2 w}{dx^2} \int_A z^2 dA =$$

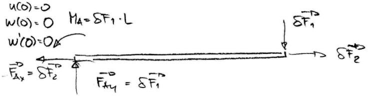
$$= -EI \frac{d^2 w}{dx^2}$$

Relabel δW to $\delta \Pi$

$$\delta \Pi = \int_0^L EA \frac{du}{dx} \frac{d\delta u}{dx}\, dx + \int_0^L EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2}\, dx - \int_0^L q_1 \delta w\, dx - \int_0^L q_2 \delta u\, dx$$

$$- F_1 \delta w(L) - F_2 \delta u(L)$$

Next, let we suppose that the beam is loaded by the virtual forces $\delta \vec{F}_1$, $\delta \vec{F}_2$, which are in equilibrium with the reactions $\vec{F}_{Ax}$, $\vec{F}_{Ay}$ a $\vec{M}_A$



Hamilton's principle: Examples

$$\delta W_E^* = -\left[\delta F_1 w(L) + \delta F_2 u(L) - \delta F_1 w(0) + (\delta F_1 L)\left(\frac{dw}{dx}\right)_{x=0} - \delta F_2 u(0)\right] =$$

$$= -\left[\delta F_1 w(L) + \delta F_2 u(L)\right]$$

$\varepsilon_{11} = \varepsilon_{xx}^{(0)} + z\,\varepsilon_{xx}^{(1)}$, $\varepsilon_{13} = \varepsilon_{xz}^{(0)}$ ⟵ Let we assume defromation in the form

$$\delta W_I^* = \int_V (\varepsilon_{11}\delta\sigma_{11} + 2\varepsilon_{13}\delta\sigma_{13})\,dV = \int_0^L \int_A \left[\left(\varepsilon_{xx}^{(0)} + z\,\varepsilon_{xx}^{(1)}\right)\delta\sigma_{xx} + 2\varepsilon_{xz}^{(0)}\delta\sigma_{xz}\right] dA\,dx =$$

$$= \int_0^L \varepsilon_{xx}^{(0)} \int_A \delta\sigma_{xx}\,dA\,dx + \int_0^L \varepsilon_{xx}^{(1)} \int_A z\,\delta\sigma_{xx}\,dA\,dx + \int_0^L 2\varepsilon_{xz}^{(0)} \int_A \delta\sigma_{xz}\,dA\,dx =$$

$$= \int_0^L \varepsilon_{xx}^{(0)}\,\delta N\,dx + \int_0^L \varepsilon_{xx}^{(1)}\,\delta M\,dx + \int_0^L 2\varepsilon_{xz}^{(0)}\,\delta V\,dx$$

$$\delta W^* = \int_0^L \left(\varepsilon_{xx}^{(0)}\delta N + \varepsilon_{xx}^{(1)}\delta M + 2\varepsilon_{xz}^{(0)}\delta V\right)dx - \left[\delta F_1 w(L) + \delta F_2 u(L)\right]$$

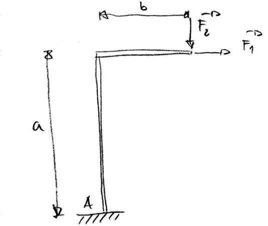
We assume the linearly elastic material again:

$$\varepsilon_{xx}^{(0)} = \frac{N}{EA}, \quad \varepsilon_{x}^{(1)} = \frac{M}{EI}, \quad 2\varepsilon_{xz}^{(0)} = \frac{VQ}{GIb}$$

$$\delta\Pi^* = \int_0^L \left(\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{VQ}{GIb}\delta V\right)dx - \left[\delta F_1 w(L) + \delta F_2 u(L)\right]$$

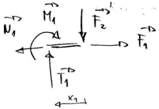
Hamilton's principle: Examples

Example 0.14: Consider the structure in the figure and express the complmentary virtual work.



$$\delta W_i^* = \int_0^L \left(\frac{N}{EA}\delta N + \frac{M}{EI}\delta M + \frac{TQ}{GIb}\delta V \right) dx$$

The internal resulting forces and moments:

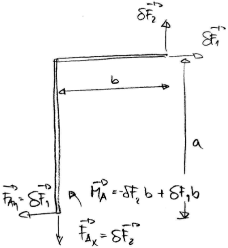


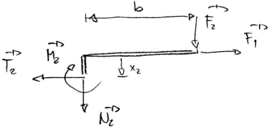
$x_1 \in (0, b)$

$N_1 = F_1$

$T_1 = F_2$

$M_1 = -F_2 \cdot x_1$





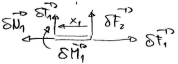
$x_2 \in (0, a)$

$N_1 = -F_2$

$T_2 = F_1$

$M_2 = -F_2 \cdot b - F_1 \cdot x_2$

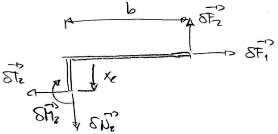
We introduce the virtual loadings, which are in the equilibrium.
The intenral resukting forces and moments



$x_1 \in (0, b)$

$\delta N_1 = \delta F_1, \quad \delta T_1 = -\delta F_2, \quad \delta M_1 = -\delta F_2 \cdot x_1$

Hamilton's principle: Examples



$x_2 \in (0, a)$

$\delta N_2 = \delta F_2 \;,\; \delta T_2 = \delta F_1 \;,\; \delta M_2 = \delta F_2 b - \delta F_1 \cdot x_2$

The complementary strain energy:

$$\delta U_1^* = \int_0^b \frac{N_1}{EA}\delta N_1 \, dx_1 + \int_0^b \frac{M_1}{EI}\delta M_1 \, dx_1 + \int_0^b \frac{T_1 Q}{GIb'}\delta T_1 \, dx_1 = \int_0^b \frac{F_1}{EA}\cdot \delta F_1 \, dx_1 + \int_0^b \frac{-F_2 x_1}{EI}\cdot(-\delta F_2 x_1)\,dx_1$$

$$+ \int_0^b \left(-\frac{F_2 Q}{GIb}\delta F_2\right)dx_1 = \frac{F_1 b}{EA}\delta F_1 + \frac{F_2 b^3}{3EI}\delta F_2 - \frac{F_2 Q}{GIb'} b\,\delta F_2 = \frac{F_1 b}{EA}\delta F_1 + \left(\frac{b^3}{3EI} - \frac{Qb}{GIb'}\right)F_2\,\delta F_2$$

kde $A = \int_A dA \;,\; I = \int_A z^2 dA \;,\; Q = \int_{A'} z\,dA \;,\; G = \frac{E}{2(1+\nu)}$

$$\delta U_2^* = \int_0^a \frac{N_2}{EA}\delta N_2 \, dx_2 + \int_0^a \frac{M_2}{EI}\delta M_2 \, dx_2 + \int_0^b \frac{T_2 Q}{GIb'}\delta T_2 \, dx_2 = \int_0^a \frac{-F_2}{EA}\delta F_2 \, dx_2 + \int_0^a \frac{-F_2 b - F_1 x_2}{EI}\cdot$$

$$\cdot(\delta F_2 b - \delta F_1 x_2)\,dx_2 + \int_0^a \frac{F_1 Q}{GIb'}\delta F_1 \, dx_2 = \frac{-F_2 a}{EA}\delta F_2 + \frac{1}{EI}\int_0^a \left(-F_2 b^2 \delta F_2 - F_1 b x_2 \delta F_2 + \right.$$

$$\left. + F_2 b x_2 \delta F_1 + F_1 x_2^2 \delta F_1\right) dx_2 + \frac{F_1 Q a}{GIb'}\delta F_1 = -\frac{F_2 a}{EA}\delta F_2 + \frac{1}{EI}\left(-F_2 b^2 a\,\delta F_2 - \frac{1}{2}F_1 b a^2 \delta F_2 + \right.$$

$$\left. + \frac{1}{2}F_2 b a^2 \delta F_1 + F_1 \frac{1}{3}a^3 \delta F_1\right) + \frac{F_1 Q a}{GIb'}\delta F_1 =$$

Hamilton's principle: Examples

$$= \left[-\frac{F_2 a}{EA} + \frac{1}{EI}\left(-F_2 b^2 a - \frac{1}{2} F_1 b a^2 \right) \right] \delta F_2 + \left[\frac{1}{EI}\left(\frac{1}{2} F_2 b a^2 + \frac{1}{3} F_1 a^3 \right) + \frac{F_1 Q a}{GIb'} \right] \delta F_1$$

The total complementary strain energy of the beam

$$\delta W_I^* = \delta U_1^* + \delta U_2^*$$

The external virtual work

$$\delta W_E^* = -\left(w\,\delta F_2 + u \cdot \delta F_1 \right).$$

Thank you!