Continuum mechanics

Lecture 1

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Kinetics - Stress at Point

Forces acting on a body can be classified as *internal* and *external*. The external forces can be classified as *body forces* and *surface forces*. Body forces act on the distribution of mass inside the body (gravitational force, magnetic forces). Body forces are usually measured per unit mass or unit volume of the body. Surface forces are contact forces acting on the boundary surface body. Example of surface forces are provided by applied forces on the surface of the body. Surface forces are reckoned per unit area. *The traction* or *the stress vector* at a point P on ΔS is defined by

$$\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\boldsymbol{f}}{\Delta S},$$

where $\boldsymbol{f}$ is the surface force acting on a small portion ΔS of the surface area S of the body Ω.

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The magnitude and direction of $\boldsymbol{t} \equiv \boldsymbol{t}^{(\boldsymbol{n})}$ depend on the orientation of ΔS given by its normal $\boldsymbol{n}$. *The normal stress* is component of $\boldsymbol{t}$ that is in the direction of $\boldsymbol{n}$, *the shear stress* is the component of $\boldsymbol{t}$ that is normal to $\boldsymbol{n}$. Let $\boldsymbol{t}^{(i)}$ denote the stress vector at point P on a plane perpendicular to x_i-axis, then

$$\boldsymbol{t}^{(i)} = \sigma_{ij}\boldsymbol{e}_j \quad (i, j = 1, 2, 3),$$

where σ_{ij} denotes the component of stress vector $\boldsymbol{t}^{(i)}$ along the x_j-direction.

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The relationship between the stress vector $\boldsymbol{t}^{(\boldsymbol{n})}$ acting on a plane given by normal $\boldsymbol{n}$ and the stress vectors on three planes perpendicular to the coordinate axes x_j is called *the Cauchy stress formula*

$$t_j^{(\boldsymbol{n})} = n_i \sigma_{ij},$$

written in matrix form as

$$\begin{bmatrix} t_1^{(\boldsymbol{n})} \\ t_2^{(\boldsymbol{n})} \\ t_3^{(\boldsymbol{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^T \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},$$

where $[\cdot]^T$ denotes the transpose of the matrix.

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The normal and shearing components of the stress vector $\boldsymbol{t}^{(\boldsymbol{n})}$ are given by

$$t_N = \boldsymbol{t}^{(\boldsymbol{n})} \cdot \boldsymbol{n} = \sigma_{ij} n_i n_j,$$
$$t_S = \sqrt{\left|\boldsymbol{t}^{(\boldsymbol{n})}\right|^2 - t_N^2}.$$

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Consider two sets of orthogonal coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3). Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ and $(\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$ denote the sets of basis vectors in the two coordinate systems. Suppose that the two coordinate systems are related by the transformation equation

$$\boldsymbol{e}'_i = a_{ij}\boldsymbol{e}_j,$$

where a_{ij} denote the directions cosines

$$a_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j.$$

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We note that the *stress dyadic*

$$\boldsymbol{\sigma} = \boldsymbol{e}_i \sigma_{ij} \boldsymbol{e}_j$$

at a point is invariant and it does not depend on the coordinate system. But the components σ_{ij} do depend on the coordinate system.

$$\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \sigma'_{ij} \boldsymbol{e}'_i \boldsymbol{e}'_j$$

and

$$(\sigma_{ij} - \sigma'_{kl} a_{ki} a_{lj}) \boldsymbol{e}_i \boldsymbol{e}_j = 0.$$

From which follows

$$\sigma_{ij} = \sigma'_{kl} a_{ki} a_{lj}.$$

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The maximum and minimum normal stresses, called *principal stresses*, act in a plane on which the shear stress is zero

$$\boldsymbol{t}^{(\boldsymbol{n})} = \lambda \boldsymbol{n},$$

where λ denotes the magnitude of the normal stress and $\boldsymbol{n}$ denotes the unit normal to the plane on which the maximum stress is acting,

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{\sigma} \cdot \boldsymbol{n}.$$

Equating these two equations, we obtain

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} - \lambda \boldsymbol{n} = 0$$

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The equation

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} - \lambda \boldsymbol{n} = 0$$

can be rewritten to the component form

$$(\sigma_{ij} - \lambda \delta_{ij}) n_j = 0,$$

where δ_{ij} is a *Kronecker delta*

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \quad .$$

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The equation

$$(\sigma_{ij} - \lambda\delta_{ij})n_j = 0,$$

has a nontrivial solution $(n_i \neq 0, i = 1, 2, 3)$ only if the determinant of the coefficient matrix is zero

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0.$$

The determinant leads to the polynomial called *characteristic* and the three values of λ are called *characteristic values* or *eigenvalues* and associated (unit) normal vectors $\boldsymbol{n}$ are called *characteristic vectors* or *eigenvectors*.

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The characteristic polynomial is of the form

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0,$$

where I_1, I_2 and I_3 are stress invariants defined as

$$\begin{aligned} I_1 &= \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}, \\ I_2 &= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2), \\ I_3 &= \det[\boldsymbol{\sigma}] = |\sigma_{ij}|. \end{aligned}$$

For symmetric tress dyadic $\boldsymbol{\sigma}$, i.e. $\sigma_{ij} = \sigma_{ji}$, the eigenvalues are real.

Equations of Equilibrium

The equation of motion for solids can be derived from the principle of of conservation of linear momentum providing the Newton's third law of action and reaction governs the internal forces. The total momentum is given by

$$\int_\Omega \rho \frac{\partial \boldsymbol{u}}{\partial t} d\Omega.$$

The time rate of the momentum is

$$\frac{d}{dt}\int_\Omega \rho \frac{\partial \boldsymbol{u}}{\partial t} d\Omega = \int_\Omega \frac{d}{dt}(\rho d\Omega)\frac{\partial \boldsymbol{u}}{\partial t} + \int_\Omega \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} d\Omega,$$

where d/dt denotes the total (material) time derivative. The first integral on the right-hand side is equal to zero because of the principle of conservation of mass of a given material

$$\frac{d}{dt}(\rho d\Omega) = 0.$$

Equations of Equilibrium

The momentum principle can now be expressed as

$$\int_{\Omega} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} d\Omega = \int_{\Omega} \boldsymbol{f} d\Omega + \int_{S} \boldsymbol{t} dS,$$

where the external forces $\boldsymbol{f}$ and $\boldsymbol{t}$ are the body force and surface tranction, respectively. Using the above derived equation

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{\sigma} \cdot \boldsymbol{n},$$

it is obtained

$$\int_{\Omega} \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} d\Omega - \int_{\Omega} \boldsymbol{f} d\Omega - \int_{S} \boldsymbol{\sigma} \cdot \boldsymbol{n} dS = 0.$$

Equations of Equilibrium

Using the divergence theorem, the surface integral can be converted to a volume integral

$$-\int_{\Omega}\left(\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}-\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}\right)d\Omega=0.$$

The equation should hold for arbitrary region Ω. This implies that the integrand of the left-hand side should be identically equal to zero

$$\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}.$$

Equations of Equilibrium

In a rectangular cartesian coordinate system, this equation takes the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

The equation of equilibrium are obtained by setting the right-hand side equal to zero

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$$

or

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}.$$

Equations of Equilibrium

One can establish the symmetry of the stress tensor using the Newton's second law for moments, under the condition that a body is not subjected to distributed couples

$$\epsilon_{ijk}\sigma_{ij} = 0 \quad \text{for } k = 1, 2, 3,$$

where ϵ_{ijk} is the *permutation symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are in cyclic order and } i \neq j \neq k, \\ -1 & \text{if } ijk \text{ are not in cyclic order and } i \neq j \neq k, \\ 0 & \text{if any of } ijk \text{ are repeated.} \end{cases}$$

Equations of Equilibrium

Or rewritten to the components

$$\begin{aligned} k = 1: &\quad \sigma_{23} - \sigma_{32} = 0 \\ k = 2: &\quad \sigma_{31} - \sigma_{13} = 0 \\ k = 3: &\quad \sigma_{12} - \sigma_{21} = 0. \end{aligned}$$

Thus, there are only six stress components that are independent.

Equations of Equilibrium

Because the equation

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}$$

contains three equations relating six stress components σ_{ij}, the equations of equilibrium are not sufficient for the determination of the stress components σ_{ij}. Additional equations are required, the strain-displacement equations and constitutive equations.

Kinetics - Stress at a Point: Examples

Example 1:

Consider the transformation from the rectangular cartesian system (x,y,z) to the cylindrical coordinate system (r,Θ,z):

$$\left.\begin{aligned} \underline{e}_r &= \underline{e}_x \cos\Theta + \underline{e}_y \sin\Theta \\ \underline{e}_\Theta &= -\underline{e}_x \sin\Theta + \underline{e}_y \cos\Theta \\ \underline{e}_z &= \underline{e}_z \end{aligned}\right\} \Leftrightarrow \begin{aligned} \underline{e}_r &= (\cos\Theta, \sin\Theta, 0) \\ \underline{e}_\Theta &= (-\sin\Theta, \cos\Theta, 0) \\ \underline{e}_z &= (0, 0, 1) \end{aligned}$$

$$\Rightarrow \underline{A} = [a_{ij}] = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \sigma'_{ij} = a_{ki} a_{lj} \sigma_{kl} \Leftrightarrow \underline{\sigma}' = \underline{A}\,\underline{\sigma}\,\underline{A}^T$$

$$\Rightarrow \begin{bmatrix} \sigma_{rr} & \sigma_{r\Theta} & \sigma_{rz} \\ \sigma_{\Theta r} & \sigma_{\Theta\Theta} & \sigma_{\Theta z} \\ \sigma_{zr} & \sigma_{z\Theta} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} \sigma_{rr} &= \sigma_{xx}\cos^2\Theta + (\sigma_{xy} + \sigma_{yx})\cos\Theta\sin\Theta + \sigma_{yy}\sin^2\Theta \\ \sigma_{r\Theta} &= (\sigma_{yy} - \sigma_{xx})\cos\Theta\sin\Theta + \sigma_{xy}\cos^2\Theta - \sigma_{yx}\sin^2\Theta \\ \sigma_{\Theta r} &= (\sigma_{yy} - \sigma_{xx})\cos\Theta\sin\Theta - \sigma_{xy}\sin^2\Theta + \sigma_{yx}\cos^2\Theta \end{aligned}$$

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$$\sigma_{\theta\theta} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - (\sigma_{xy} + \sigma_{yx}) \cos\theta \sin\theta$$

$$\sigma_{rz} = \sigma_{xz} \cos\theta + \sigma_{yz} \sin\theta$$

$$\sigma_{\theta z} = -\sigma_{xz} \sin\theta + \sigma_{yz} \cos\theta$$

$$\sigma_{zr} = \sigma_{zx} \cos\theta + \sigma_{zy} \sin\theta$$

$$\sigma_{z\theta} = -\sigma_{zx} \sin\theta + \sigma_{zy} \cos\theta$$

$$\sigma_{zz} = \sigma_{zz}$$

Assuming $\sigma_{ij} = \sigma_{ji}$:

$$\sigma_{rr} = \sigma_{xx} \cos^2\theta + 2\sigma_{xy} \cos\theta \sin\theta + \sigma_{yy} \sin^2\theta$$

$$\sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx}) \cos\theta \sin\theta + \sigma_{xy} (\cos^2\theta - \sin^2\theta)$$

$$\sigma_{\theta\theta} = \sigma_{xx} \sin^2\theta - 2\sigma_{xy} \cos\theta \sin\theta + \sigma_{yy} \cos^2\theta$$

$$\sigma_{rz} = \sigma_{xz} \cos\theta + \sigma_{yz} \sin\theta$$

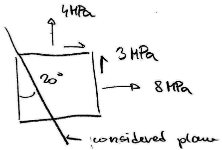
$$\sigma_{\theta z} = -\sigma_{xz} \sin\theta + \sigma_{yz} \cos\theta$$

$$\sigma_{zz} = \sigma_{zz}$$

Kinetics - Stress at a Point: Examples

Example 2:

Let evaluate normal and shear stress component in the marked plane



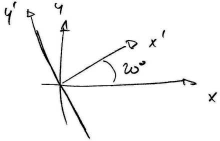
$\sigma_{xx} = 8\,MPa$

$\sigma_{yy} = 4\,MPa$

$\sigma_{xy} = 3\,MPa$

$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$

$$\Rightarrow \quad \underline{\sigma} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



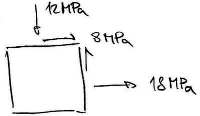
$$\Rightarrow \quad \underline{\sigma}' = \begin{bmatrix} \cos 20^\circ & \sin 20^\circ \\ -\sin 20^\circ & \cos 20^\circ \end{bmatrix} \begin{bmatrix} 8 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix} =$$

$$= \begin{bmatrix} 9.46 & 1.01 \\ 1.01 & 2.54 \end{bmatrix} \Rightarrow \quad \begin{aligned} \sigma'_{xx} &= 9.46\,MPa \\ \sigma'_{xy} &= 1.01\,MPa \end{aligned}$$

Kinetics - Stress at a Point: Examples

Example 3:

let find characteristic values (principal stresses) and vectors (principle planes) of the stress tensor



$$\Rightarrow \underline{S} = \begin{bmatrix} 18 & 8 & 0 \\ 8 & -12 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 8 & -12 \end{bmatrix}$$

Characteristic polynomial: $-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$

$$I_1 = \sigma_{xx} + \sigma_{yy} = 18 - 12 = 6$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = -18 \cdot 12 - 8^2 = -280$$

$$I_3 = 0$$

$$\Downarrow$$

$$-\lambda^3 + 6\lambda^2 + 280\lambda = 0 \Rightarrow \begin{cases} \lambda_1 = 0 \text{ MPa} \\ \lambda_2 = \dfrac{6 + \sqrt{6^2 - 4(-280)}}{2} = 20 \text{ MPa} \\ \lambda_3 = \dfrac{6 - \sqrt{6^2 - 4(-280)}}{2} = -14 \text{ MPa} \end{cases}$$

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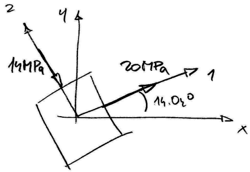
$\underline{n_1}$: $\begin{bmatrix} 18-20 & 8 \\ 8 & -12-20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 8 \\ 8 & -32 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow -x + 4y = 0 \Rightarrow x = 4y$

Choosing $y = 1$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow \underline{n_1} = \frac{1}{\sqrt{4^2+1^2}} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.97 \\ 0.24 \end{bmatrix} = \begin{bmatrix} \cos 14.04^\circ \\ \sin 14.04^\circ \end{bmatrix}$

$\underline{n_2}$: $\begin{bmatrix} 18+14 & 8 \\ 8 & -12+14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 32 & 8 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 4x + y = 0 \Rightarrow x = -\frac{1}{4} y$

Choosing $y = 1$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \Rightarrow \underline{n_2} = \frac{1}{\sqrt{(\frac{1}{4})^2+1^2}} \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.24 \\ 0.97 \end{bmatrix} = \begin{bmatrix} -\sin 14.04^\circ \\ \cos 14.04^\circ \end{bmatrix}$



Thank you!