

Continuum mechanics

Lecture 2

Tomáš Profant

Brno University of Technology
Faculty of Mechanical Engineering

Kinematics - Strain at Point

A measure of the deformation in a body is provided by the change in the distance between points in the body. Consider two neighboring points

$$P = (x_1, x_2, x_3) \text{ and } Q = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$$

in the reference configuration. Under the action of externally applied forces, the body deforms and the points P and Q move to new places

$$\bar{P} = (\xi_1, \xi_2, \xi_3) \text{ and } \bar{Q} = (\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3).$$

Since points P and Q are arbitrary, the discussion applies to any point in the body.

Kinematics - Strain at Point

The motion of a particle occupying position \boldsymbol{x} in the undeformed body to point $\boldsymbol{\xi}$ in the deformed body can be expressed by the transformation

$$\xi_i = \xi_i(x_1, x_2, x_3, t),$$

where t denotes time. By assumption a continuous medium cannot have gaps or overlaps. Therefore, a one-to-one correspondence exists between points in the undeformed body and points in the deformed body. Consequently, a unique inverse to $\boldsymbol{\xi}$ holds

$$x_i = x_i(\xi_1, \xi_2, \xi_3, t).$$

Kinematics - Strain at Point

If the distances between points P and Q is ds and the distance between points \bar{P} and \bar{Q} is $d\bar{s}$, then the measure of deformation of the body is

$$(d\bar{s})^2 - (ds)^2 = d\xi_i d\xi_i - dx_i dx_i = 2\varepsilon_{ij} dx_i dx_j,$$

where ε_{ij} are the components of the *Green strain tensor* at point P

$$\varepsilon_{ij} = \left(\frac{\partial \xi_m}{\partial x_i} \frac{\partial \xi_m}{\partial x_j} - \delta_{ij} \right) dx_i dx_j.$$

The symbol δ_{ij} is a Kronecker delta.

Kinematics - Strain at Point

The displacements can be written as

$$u_m = \xi_m - x_m \Rightarrow \frac{\partial \xi_m}{\partial x_i} = \frac{\partial u_m}{\partial x_i} + \delta_{mi}.$$

The substitution of this expression into

$$\varepsilon_{ij} = \left(\frac{\partial \xi_m}{\partial x_i} \frac{\partial \xi_m}{\partial x_j} - \delta_{ij} \right) dx_i dx_j$$

gives the expression of strains in terms of the displacements at point P

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_j} \frac{\partial u_m}{\partial x_i} \right).$$

Kinematics - Strain at Point

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_j} \frac{\partial u_m}{\partial x_i} \right).$$

We will assume in the following that the displacement components are small compared to unity, i.e.

$$\frac{\partial u_i}{\partial x_j} \ll 1, \quad \left(\frac{\partial u_i}{\partial x_j} \right)^2 \approx 0,$$

then strain components ε_{ij} become the *infinitesimal strain components*

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Kinematics - Strain at Point

Similarly to the stresses at a point, it is a matter of interest to know the components of strain at a point in one coordinate system, e.g. (x'_1, x'_2, x'_3) , if they are known in the another coordinate system (x_1, x_2, x_3) at the same point. It is not difficult to proof, that the tensor character of strain is associated with the transformation relation

$$\varepsilon'_{ij} = a_{im}a_{jn}\varepsilon_{mn},$$

where

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j.$$

Compatibility Conditions

When the strain components are given, the determination of the displacements is not always possible, because there are six strain components related to three displacement components. There are six differential equations involving three unknowns. Thus, the six equations should be compatible with each other in the sense that any three equations should give the same displacement field. Assuming the infinitesimal strain components, the only way, how to connect six partial differential equation

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

and eliminate the components of displacements appearing in them, is to derive them two times. Then it can be observed

$$\frac{\partial^2 e_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 e_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 e_{lj}}{\partial x_k \partial x_i} + \frac{\partial^2 e_{ki}}{\partial x_l \partial x_j}.$$

Compatibility Conditions

The equation

$$\frac{\partial^2 e_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 e_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 e_{lj}}{\partial x_k \partial x_i} + \frac{\partial^2 e_{ki}}{\partial x_l \partial x_j}.$$

forms the necessary and sufficient conditions for the existence of a single-valued displacement field (when the strains are given). It is 81 equations, but only six of them are non trivial and different and linearly independent from each other.

Compatibility Conditions

These compatibility equations are

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2 \frac{\partial^2 e_{12}}{\partial x_1 \partial x_2},$$

$$\frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} = 2 \frac{\partial^2 e_{23}}{\partial x_2 \partial x_3},$$

$$\frac{\partial^2 e_{11}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_1^2} = 2 \frac{\partial^2 e_{13}}{\partial x_1 \partial x_3},$$

$$\frac{\partial}{\partial x_1} \left(-\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} \right) = \frac{\partial^2 e_{11}}{\partial x_2 \partial x_3},$$

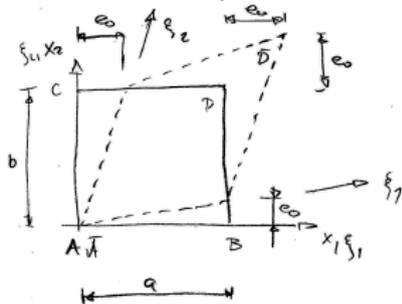
$$\frac{\partial}{\partial x_2} \left(-\frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} \right) = \frac{\partial^2 e_{22}}{\partial x_1 \partial x_3},$$

$$\frac{\partial}{\partial x_3} \left(-\frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{13}}{\partial x_2} \right) = \frac{\partial^2 e_{33}}{\partial x_1 \partial x_2}.$$

Kinematics - Strain at Point: Examples

Example 4:

Consider a rectangular block of dimensions $a \times b \times h$, $h \ll a, b$. The block is deformed to the diamond shape. Let express the components of strain tensor.



$(f_1, f_2, 0)$ - coordinates of a material point in the undeformed configuration
 $(x_1, x_2, 0)$ - spatial coordinates of the point in physical space

$$x_1 = f_1 + \frac{e_0}{b} f_2$$

$$x_2 = f_2 + \frac{e_0}{a} f_1$$

$$x_3 = f_3 \approx 0$$

Then the displacement components of the material point are

$$u_1 = x_1 - f_1 = \frac{e_0}{b} f_2, \quad u_2 = x_2 - f_2 = \frac{e_0}{a} f_1, \quad u_3 = 0$$

An undeformed body is for $f_i = x_i$. The strains $\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)$:

$$\epsilon_{11} = \frac{1}{2} \left(\frac{e_0}{a} \right)^2, \quad \epsilon_{12} = \frac{e_0}{2b} + \frac{e_0}{2a}, \quad \epsilon_{22} = \frac{1}{2} \left(\frac{e_0}{b} \right)^2$$

Kinematics - Strain at Point: Examples

For infinitesimal strain components, one gets

$$\epsilon_{11} = 0, \epsilon_{22} = 0, \epsilon_{12} = \frac{e_0}{2b} + \frac{e_0}{2a}$$

Now let observe the strain ϵ'_{11} in the direction \bar{AD} :

$$\epsilon'_{11} = a_{i1} a_{j1} \epsilon_{ij} = a_{11} a_{11} \epsilon_{11} + a_{11} a_{12} \epsilon_{12} + a_{12} a_{11} \epsilon_{21} + a_{12} a_{22} \epsilon_{22}$$

where

$$\epsilon_{21} = \epsilon_{12}, \quad a_{11} = \cos \theta, \quad a_{12} = \sin \theta, \quad a_{21} = -\sin \theta, \quad a_{22} = \cos \theta.$$

Then

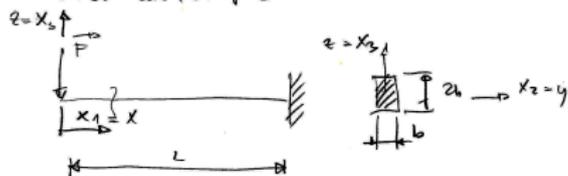
$$\epsilon'_{11} = \cos^2 \theta \epsilon_{11} + 2 \cos \theta \sin \theta \epsilon_{12} + \sin^2 \theta \epsilon_{22}$$

$$\begin{aligned} \cos \theta &= \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \theta = \frac{b}{\sqrt{a^2 + b^2}}, \quad \Rightarrow \quad \epsilon'_{11} = \frac{a^2}{a^2 + b^2} \cdot \frac{1}{2} \left(\frac{e_0}{a} \right)^2 + 2 \frac{ab}{a^2 + b^2} \left(\frac{e_0}{2b} + \frac{e_0}{2a} \right) \\ &\quad + \frac{b^2}{a^2 + b^2} \cdot \frac{1}{2} \left(\frac{e_0}{b} \right)^2 \\ &= \frac{e_0^2 + e_0(a+b)}{a^2 + b^2} \end{aligned}$$

Kinematics - Strain at Point: Examples

Example 5:

Consider an isotropic cantilever beam as it is in fig. Express the compatibility conditions.



The equilibrium equation is (omitting volume forces)

$$\begin{aligned}
 EI \frac{d^4 w}{dx^4} &= 0 \rightarrow EI w'''' = C_1 \\
 EI w'' &= C_1 x + C_2 \\
 EI w' &= \frac{1}{2} C_1 x^2 + C_2 x + C_3 \\
 EI w &= \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4
 \end{aligned}$$

Boundary conditions are: $x=0: EI w'' = 0, EI w'''' = -F$
 $x=L: w = 0, w' = 0$

From this follows

$$C_1 = -F, C_2 = 0, C_3 = \frac{1}{2} FL^2, C_4 = \frac{1}{6} FL^3 - \frac{1}{2} FL^3 = -\frac{1}{3} FL^3$$

and

$$EI w = -\frac{1}{6} F x^3 + \frac{1}{2} FL^2 x - \frac{1}{3} FL^3$$

Kinematics - Strain at Point: Examples

$$u = -z \frac{dw}{dx} \Rightarrow EI u = -\frac{1}{2} F z x^2 - \frac{1}{2} F L^2 z$$

From elementary beam theory is known - $V = -F$, $\mu = \frac{E}{2(1+\nu)}$, $Q = b(h-z) \frac{1}{2}(h+z)$ - then

$$\begin{aligned} \epsilon_{11} = \frac{du}{dx} &= \frac{Fzx}{EI}, & \epsilon_{22} = -\nu \epsilon_{11} &= -\frac{\nu Fzx}{EI}, & \epsilon_{12} &= \frac{\sigma_{12}}{\mu} = \frac{2(1+\nu)}{E} \frac{VQ}{Ib} \\ & & & & &= \frac{2(1+\nu)}{E} \frac{-F}{b} b(h-z) \frac{1}{2}(h+z) \\ & & & & &= -\frac{1+\nu}{E} F(h^2 - z^2) \end{aligned}$$

For plane problem, the compatibility equations reduce to one

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

from which

$0 = 0 \Rightarrow$ A compatibility condition is fulfilled.

Thank you!