

Continuum mechanics

Lecture 9

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Hamilton's principle is the most general and basic principle of mechanics, from which the Euler-Lagrange equations and other laws of mechanics, as well as continuum mechanics, simply follow. The body involving no motion or forces applied sufficiently slowly that the motion is independent of time and the inertia forces are negligible is considered in the following. A wide range of continuum mechanics problems can be formulated so that this condition is satisfied. The assumption of time-independent motion also allows us to introduce several important variational methods capable of finding solutions to fairly general continuum mechanics problems.

Unit-Dummy-Displacement Method

The principle of virtual work can be reduced to the form in the case of quasi-static loading process

$$\int_{\Omega} \mathbf{f} \cdot \delta \mathbf{u} d\Omega + \int_{S_2} \hat{\mathbf{t}} \cdot \delta \mathbf{u} dS - \int_{\Omega} \boldsymbol{\sigma} : \delta \mathbf{e} d\Omega = 0,$$

can be used to directly determine reaction forces and displacements in structural problems. Consider the reaction force (or moment) \mathbf{R}_0 at point 0 such, that

$$\mathbf{R}_0 = R_0 \mathbf{e}_0,$$

where \mathbf{e}_0 is the unit vector in the direction of the reaction \mathbf{R}_0 . We prescribe a virtual displacement (or rotation) $\delta \mathbf{u}_0$ as

$$\delta \mathbf{u}_0 = \delta u_0 \mathbf{e}_0$$

at the point 0 in the elastic structure, but keep all other external forces on the structure stationary.

Unit-Dummy-Displacement Method

The virtual strains δe_{ij}^0 owing to the virtual displacement δu_0 in the direction e_0 are determined from the kinematic considerations. Then the method of virtual work reads as

$$\int_{\Omega} f_i \delta u_i^0 d\Omega + R_0 \delta u_0 = \int_{\Omega} \delta U_0 d\Omega = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega$$

or for $\mathbf{f} = 0$ as

$$R_0 \delta u_0 = \int_{\Omega} \delta U_0 d\Omega = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega$$

where σ_{ij} are the actual stresses and δe_{ij}^0 or δu_i^0 are the virtual strains or displacements, respectively, of the entire structure, consistent with the geometric constraints.

Unit-Dummy-Displacement Method

Since δu_0 is arbitrary, one can take $\delta u_0 = 1$ and δe_{ij}^0 or δu_i^0 denote the virtual strains or displacements, respectively, corresponding to the unit displacement at point 0. This procedure is called the *unit-dummy-displacement method* and for the magnitude of the reaction R_0 we get

$$R_0 = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega - \int_{\Omega} f_i \delta u_i^0 d\Omega$$

or if the volume loading f is zero-valued

$$R_0 = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega.$$

Unit-Dummy-Load Method

The basic idea can be described in analogy with the unit-dummy-displacement method. If the true displacement \mathbf{u}_0 in the direction of the unit vector \mathbf{e}_0 , i.e.

$$\mathbf{u}_0 = u_0 \mathbf{e}_0,$$

is at the point 0 of an elastic structure, we can prescribe a virtual force $\delta \mathbf{R}_0$ at that point and the same direction

$$\delta \mathbf{R}_0 = \delta R_0 \mathbf{e}_0.$$

Unit-Dummy-Load Method

The application of virtual force induces a system of virtual stresses $\delta\sigma_{ij}$ that *satisfy* the equilibrium equations. Then, instead of external work V and potential U , their complementary counterparts V^* and U^* as the complementary potential energy Π^* is used in the Hamilton's principle. Then the principle of total complementary virtual work can be derive as follows

$$\delta\Pi^* = 0 \Rightarrow \delta V^* + \delta U^* = 0.$$

Unit-Dummy-Load Method

Consider

$$\delta V^* = -\mathbf{u}_0 \cdot \delta \mathbf{R}_0 = -u_0 \delta R_0,$$

we have

$$u_0 \delta R_0 = \int_{\Omega} \delta U_0^* d\Omega = \int_{\Omega} e_{ij} \delta \sigma_{ij}^0 d\Omega.$$

Once again we can take $\delta R_0 = 1$ and calculate corresponding virtual internal stresses $\delta \sigma_{ij}^0$ from equilibrium equations. From the previous equation consequently can be evaluated the magnitude of the true displacement \mathbf{u}_0 as follows

$$u_0 = \int_{\Omega} e_{ij} \delta \sigma_{ij}^0 d\Omega.$$

Castigliano's First Theorem

Consider a general three-dimensional structure that is in equilibrium under the action of N forces

$$\mathbf{F}_i = F_i \mathbf{e}_i \quad \text{for } i = 1, 2, \dots, N,$$

where \mathbf{e}_i are unit vectors. Let

$$\mathbf{u}_i = u_i \mathbf{e}_i + u_i^\perp \mathbf{e}_i^\perp \quad \text{for } i = 1, 2, \dots, N$$

be the displacement corresponding to the forces \mathbf{F}_i .

Castigliano's First Theorem

The unit vector e_i^\perp is orthogonal to the unit vector e_i so the displacement component u_i^\perp does not contribute to the potential energy of the external forces V , which is equal to

$$V = - \left(\sum_{i=1}^N F_i u_i + \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\Omega \right),$$

because the displacements u_i are points values, not functions of positions. It is also worth to note that the previous procedure can be used for moments M_i and angles θ_i .

Castigliano's First Theorem

We assume that the displacement functions $\mathbf{u} = \mathbf{u}(u_1, u_2, \dots, u_N)$ of the body can be expressed in terms of u_i . Therefore, u_i serve as the generalized coordinates and the strain energy U of the body can be expressed in terms of u_i , $i = 1, 2, \dots, N$. The total potential energy of the body is given by

$$\Pi = U + V = U(u_1, u_2, \dots, u_N) - \sum_{i=1}^N \left(F_i u_i + \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\Omega \right).$$

Castigliano's First Theorem

For any virtual variation δu_i in the displacement u_i , the variation $\delta\Pi$ in the total potential energy Π must vanish

$$\begin{aligned}\delta\Pi &= \frac{\partial U}{\partial u_i} \delta u_i - F_i \delta u_i - \int_{\Omega} \mathbf{f} \cdot \frac{\partial \mathbf{u}}{\partial u_i} \delta u_i d\Omega \\ &= \left(\frac{\partial U}{\partial u_i} - F_i - \int_{\Omega} \mathbf{f} \cdot \frac{\partial \mathbf{u}}{\partial u_i} d\Omega \right) \delta u_i = 0 \quad \text{for } i = 1, 2, \dots, N.\end{aligned}$$

Since the variations $\delta u_1, \delta u_2, \dots, \delta u_N$ are independent of each other, it follows that

$$F_i = \frac{\partial U}{\partial u_i} - \int_{\Omega} \mathbf{f} \cdot \frac{\partial \mathbf{u}}{\partial u_i} d\Omega, \quad i = 1, 2, \dots, N$$

or for $\mathbf{f} = 0$ the simplified expression is obtained

$$F_i = \frac{\partial U}{\partial u_i}, \quad i = 1, 2, \dots, N.$$

Castigliano's First Theorem

The previous equation is the general statement of Castigliano's first theorem: *If the strain energy of a structural system can be expressed in terms of N independent displacements u_1, u_2, \dots, u_N corresponding to N specified forces F_1, F_2, \dots, F_N , the first partial derivative of the strain energy with respect to any displacement u_i (under the load F_i) is equal to the force F_i in the direction of u_i .*

Castigliano's First Theorem

The first Castigliano's theorem is a special case of the principle of virtual displacements and equivalent to the unit-dummy-displacement method if the virtual strains and displacements of the entire structure δe_{ij}^0 and δu_i^0 for unit displacement $\delta u_0 = 1$ in

$$R_0 = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega - \int_{\Omega} f_i \delta u_i^0 d\Omega$$

are differentiable with respect to u_0 and can be written to the form

$$\delta e_{ij}^0 = \left[\frac{\partial e_{ij}}{\partial u_0} \delta u_0 \right]_{\delta u_0=1} \quad \text{and} \quad \delta u_i^0 = \left[\frac{\partial u_i}{\partial u_0} \delta u_0 \right]_{\delta u_0=1} .$$

Castigliano's First Theorem

Then substituting these relations into

$$R_0 = \int_{\Omega} \sigma_{ij} \delta e_{ij}^0 d\Omega - \int_{\Omega} f_i \delta u_i^0 d\Omega$$

leads to

$$\begin{aligned} R_0 &= \int_{\Omega} \frac{\partial U_0}{\partial e_{kl}} \frac{\partial e_{kl}}{\partial u_0} d\Omega - \int_{\Omega} f_k \frac{\partial u_k}{\partial u_0} d\Omega \\ &= \frac{\partial U}{\partial u_0} - \int_{\Omega} f_k \frac{\partial u_k}{\partial u_0} d\Omega, \end{aligned}$$

which is the Castigliano's first theorem for $i = 0$.

Castigliano's Second Theorem

Contrary to the first Castigliano's theorem, the second one is based on the total complementary energy principle. If a structural system is in equilibrium under the action of N forces

$$\mathbf{F}_i = F_i \mathbf{e}_i, \quad \text{for } i = 1, 2, \dots, N,$$

where \mathbf{e}_i are unit vectors, then the same procedure as presented for the first Castigliano's theorem leads for the complementary energy $U^* = U^*(F_1, F_2, \dots, F_N)$ to the equation

$$u_i = \frac{\partial U^*}{\partial F_i}.$$

This equation represents *the Castigliano's second theorem* and u_i is the component of the displacement \mathbf{u}_i at the point of the point force \mathbf{F}_i , which is in the direction of the unit vector \mathbf{e}_i .

Castigliano's Second Theorem

A previous equation is valid for structures that are linearly elastic as well as nonlinearly elastic. When they are linearly elastic, we have $U = U^*$ and one can express the strain energy in terms of displacements $U = U(u_i)$ or forces $U = U(F_i)$. Hence, the unit-dummy-load method is equivalent to the Castigliano's second theorem.

Betti's and Maxwell's Reciprocity Theorems

The principle of superposition is said to hold for a linear elastic body if the displacements obtained under a given set of forces is equal to the sum of the individual displacements that would be obtained by applying the single forces separately. On the other hand, the principle of superposition is not valid for strain and potential energies, because they are quadratic functions of displacements or forces. In other words, when a linear elastic body is subjected to more than one external force, the total work caused by external forces is not equal to the sum of the works that are obtained by applying the single forces separately.

Betti's and Maxwell's Reciprocity Theorems

Consider a linear elastic solid that is in equilibrium under the action of two external forces F_1 and F_2 . Since the order of application of the forces is arbitrary, we suppose that force F_1 is applied first. Let W_1 be the work produced by F_1 . Then, we apply force F_2 , which produces work W_2 . This work is the same as that produced by force F_2 , if it alone were acting on the body. When force F_2 is applied, force F_1 , which is already acting on the body, does additional work, because its point of application is displaced, owing to the deformation caused by force F_2 . Let us denote this work by W_{12} . Thus the total work done by the application of forces F_1 and F_2 is

$$W = W_1 + W_2 + W_{12}.$$

Work W_{12} , which can be positive or negative, is zero if and only if the displacement of the point of application of force F_1 produced by force F_2 is zero or is perpendicular to the direction of F_1 .

Betti's and Maxwell's Reciprocity Theorems

Now we change the order of application. Then the total work done is equal to

$$\bar{W} = W_1 + W_2 + W_{21},$$

where W_{21} is the work done by force F_2 , because of the application of force F_1 . The work done in both cases should be the same because, at the end, the elastic body is loaded by the same pair of external forces. Thus we have

$$W = \bar{W} \quad \text{or} \quad W_{12} = W_{21}.$$

This equation is a mathematical statement of *the Betti's reciprocity theorem*.

Betti's and Maxwell's Reciprocity Theorems

Applied to a three-dimensional elastic body, the previous equation takes the form

$$\int_{\Omega} f_i \bar{u}_i d\Omega + \int_{S_2} t_i \bar{u}_i dS = \int_{\Omega} \bar{f}_i u_i d\Omega + \int_{S_2} \bar{t}_i u_i dS,$$

where \bar{u}_i are the displacements produced by the body forces \bar{f}_i and surface forces \bar{t}_i , and u_i are the displacements produced by body forces f_i and surface forces t_i . Hence, the Betti's reciprocity theorem states: *if a linear elastic body is subjected to two different set of forces, the work done by the first system of forces in moving through the displacements produced by the second system of forces is equal to the work done by the second system of forces in moving through the displacements produced by the first system of forces.*

Betti's and Maxwell's Reciprocity Theorems

Consider a linear elastic solid subjected to force \mathbf{F}_A of unit magnitude acting at point A and force \mathbf{F}_B of unit magnitude acting at a different point B of the body. Let \mathbf{u}_{AB} be the displacement of point A in the direction of force \mathbf{F}_A produced by unit force \mathbf{F}_B , and \mathbf{u}_{BA} be the displacement of point B in the direction of force \mathbf{F}_B produced by unit force \mathbf{F}_A . From Betti's theorem it follows that

$$\mathbf{F}_A \cdot \mathbf{u}_{AB} = \mathbf{F}_B \cdot \mathbf{u}_{BA}$$

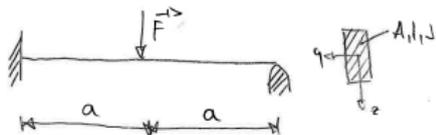
or

$$u_{AB} = u_{BA}.$$

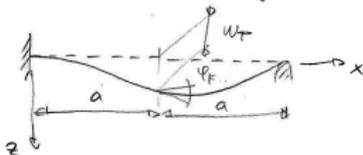
It is a statement of *Maxwell's theorem*, which states: *if the displacement u_{AB} of point A in the direction of unit force \mathbf{F}_A produced by unit force \mathbf{F}_B acting at point B is equal to the displacement u_{BA} of point B in the direction of unit force \mathbf{F}_B produced by a unit force \mathbf{F}_A acting at point A .*

Variational Methods: Examples

Example 9: Let find w_F using the unit-dummy - displ. method.



a) A construction of dummy displ. functions



$$w_1^{IV} = 0, w_2^{IV} = 0$$

$$w_1''' = c_1, w_2''' = D_1$$

$$w_1'' = c_1 x_1 + c_2, w_2'' = D_1 x_2 + D_2$$

$$w_1' = \frac{1}{2} c_1 x_1^2 + c_2 x_1 + c_3, w_2' = \frac{1}{2} D_1 x_2^2 + D_2 x_2 + D_3$$

$$w_1 = \frac{1}{6} c_1 x_1^3 + \frac{1}{2} c_2 x_1^2 + c_3 x_1 + c_4, w_2 = \frac{1}{6} D_1 x_2^3 + \frac{1}{2} D_2 x_2^2 + D_3 x_2 + D_4$$

$$x_1 = 0: w_1 = 0 \wedge w_1' = 0 \Rightarrow c_3 = c_4 = 0$$

$$x_1 = a: w_1 = w_F \wedge w_1' = \varphi_F:$$

$$w_F = \frac{1}{6} c_1 a^3 + \frac{1}{2} c_2 a^2$$

$$\varphi_F = \frac{1}{2} c_1 a^2 + c_2 a$$

$$x_2 = 0: w_2 = w_F \wedge w_2' = \varphi_F \Rightarrow D_4 = w_F, D_3 = \varphi_F$$

$$x_2 = a: w_2 = 0 \wedge w_2' = 0:$$

$$0 = \frac{1}{6} D_1 a^3 + \frac{1}{2} D_2 a^2 + \varphi_F a + w_F$$

$$0 = D_1 a + D_2$$

$$c_1 = -12a^{-3} w_F + 6a^{-2} \varphi_F$$

$$c_2 = -6a^{-2} w_F - 2a^{-1} \varphi_F$$

$$D_1 = 3a^{-3} w_F + 3a^{-2} \varphi_F$$

$$D_2 = -3a^{-2} w_F - 3a^{-1} \varphi_F$$

Variational Methods: Examples

$$w_1 = (3a^{-2}x^2 - 2a^{-3}x^3)w_F + (-a^{-1}x^2 + a^{-2}x^3)\varphi_F$$

$$w_1' = (6a^{-2}x - 6a^{-3}x^2)w_F + (-2a^{-1}x + 3a^{-2}x^2)\varphi_F$$

$$w_2 = (1 - \frac{3}{2}a^{-2}x^2 + \frac{1}{2}a^{-3}x^3)w_F + (x - \frac{3}{2}a^{-1}x^2 + \frac{1}{2}a^{-2}x^3)\varphi_F$$

$$w_2' = (-3a^{-2}x + \frac{3}{2}a^{-3}x^2)w_F + (1 - 3a^{-1}x + \frac{3}{2}a^{-2}x^2)\varphi_F$$

b) The dummy displ. functions for $w_F=1$ and $\varphi_F=1$:

$$\phi_{w_F} = w_1|_{w_F=1, \varphi_F=0} = 3a^{-2}x^2 - 2a^{-3}x^3$$

$$\phi_{\varphi_F} = w_1'|_{w_F=0, \varphi_F=1} = -2a^{-1}x + 3a^{-2}x^2$$

$$\phi_{w_F} = w_2|_{w_F=1, \varphi_F=0} = 1 - \frac{3}{2}a^{-2}x^2 + \frac{1}{2}a^{-3}x^3$$

$$\phi_{\varphi_F} = w_2'|_{w_F=0, \varphi_F=1} = 1 - 3a^{-1}x + \frac{3}{2}a^{-2}x^2$$

c) The internal virtual work done due to the unit-displacement $w_F=1$

$$\begin{aligned} \int_V \delta U|_{\delta w_F=1} dV &= \frac{1}{2} \int_0^a \delta [EI (w_1'')^2] |_{\delta w_F=1} dx_1 + \frac{1}{2} \int_0^a \delta [EI (w_2'')^2] |_{\delta w_F=1} dx_2 \\ &= \int_0^a EI w_1'' \delta w_1'' |_{\delta w_F=1} dx_1 + \int_0^a EI w_2'' \delta w_2'' |_{\delta w_F=1} dx_2 \end{aligned}$$

Variational Methods: Examples

$$\begin{aligned} &= \int_0^a EI w_1'' \left(\frac{\partial w_1''}{\partial w_F} \delta w_F \right) \Big|_{\delta w_F=1} dx_1 + \int_0^a EI w_2'' \left(\frac{\partial w_2''}{\partial w_F} \delta w_F \right) \Big|_{\delta w_F=1} dx_2 \\ &= \int_0^a EI w_1'' \phi_{w_{1F}}'' dx_1 + \int_0^a EI w_2'' \phi_{w_{2F}}'' dx_2 \end{aligned}$$

d) The internal virtual work done due to the unit-rotation $\varphi_F=1$

$$\begin{aligned} \int_V \delta U_0 \Big|_{\delta \varphi_F=1} dV &= \frac{1}{2} \int_0^a \delta [EI (w_1'')^2] \Big|_{\delta \varphi_F=1} dx_1 + \frac{1}{2} \int_0^a \delta [EI (w_2'')^2] \Big|_{\delta \varphi_F=1} dx_2 \\ &= \int_0^a EI w_1'' \delta w_1'' \Big|_{\delta \varphi_F=1} dx_1 + \int_0^a EI w_2'' \delta w_2'' \Big|_{\delta \varphi_F=1} dx_2 \\ &= \int_0^a EI w_1'' \left(\frac{\partial w_1''}{\partial \varphi_F} \delta \varphi_F \right) \Big|_{\delta \varphi_F=1} dx_1 + \int_0^a EI w_2'' \left(\frac{\partial w_2''}{\partial \varphi_F} \delta \varphi_F \right) \Big|_{\delta \varphi_F=1} dx_2 \\ &= \int_0^a EI w_1'' \phi_{\varphi_{1F}}'' dx_1 + \int_0^a EI w_2'' \phi_{\varphi_{2F}}'' dx_2 \end{aligned}$$

Variational Methods: Examples

e) Unit - dummy - displacement method:

$$(F \delta w_F) \Big|_{\delta w_F=1} = \int_V \sigma_{xx} \delta \epsilon_{xx} \Big|_{\delta w_F=1} dV$$

$$(0 \delta \varphi_F) \Big|_{\delta \varphi_F=1} = \int_V \sigma_{xx} \delta \epsilon_{xx} \Big|_{\delta \varphi_F=1} dV$$

$$F = EI \left[\int_0^a w_1'' \cdot \phi_{w_{1F}}'' dx_1 + \int_0^a w_2'' \phi_{w_{2F}}'' dx_2 \right]$$

$$0 = EI \left[\int_0^a w_1'' \phi_{\varphi_{1F}}'' dx_1 + \int_0^a w_2'' \phi_{\varphi_{2F}}'' dx_2 \right]$$

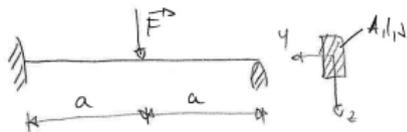
$$F = EI \left\{ \int_0^a \left[(6a^2 - 12a^3 x_1) w_F + (-2a^{-1} + 6a^2 x) \varphi_F \right] (6a^2 - 12a^3 x_1) dx_1 \right. \\ \left. + \int_0^a \left[(-3a^2 + 3a^3 x_2) w_F + (-3a^{-1} + 3a^2 x_2) \varphi_F \right] (-3a^2 + 3a^3 x_2) dx_2 \right\}$$

$$0 = \int_0^a \left[(6a^2 - 12a^3 x_1) w_F + (-2a^{-1} + 6a^2 x_1) \varphi_F \right] (6a^2 x_1) dx_1 \\ + \int_0^a \left[(-3a^2 + 3a^3 x_2) w_F + (-3a^{-1} + 3a^2 x_2) \varphi_F \right] 3a^2 dx_2$$

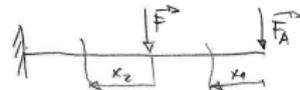
$$\frac{1}{EI} F = \left. \begin{aligned} 15a^3 w_F - 3a^2 \varphi_F \\ 0 = -3a^2 w_F + 4a^{-1} \varphi_F \end{aligned} \right\} \Rightarrow \begin{aligned} w_F &= \frac{4}{36} F a^3 \frac{1}{EI} \\ \varphi_F &= \frac{1}{32} F a^2 \frac{1}{EI} \end{aligned}$$

Variational Methods: Examples

Example 10: Consider the statically indeterminate beam under the point force F loading. Let find deflection w_F using the Castigliano II theorem.



a) the internal moments



$$\vec{H}_1 \quad \vec{F}_A \quad x_1 \in (0, a)$$

$$\vec{H}_1 = -F_A x_1$$

$$\vec{H}_2 \quad \vec{F} \quad \vec{F}_A \quad x_2 \in (0, a)$$

$$H_2 = -F_A(x_2 + a) - F \cdot x_2$$

b) F_A evaluation using Castigliano II theorem

$$0 = \frac{\partial U^*}{\partial F_A}$$

$$0 = \int_0^a \frac{\partial}{\partial F_A} \left(\frac{H_1^2}{2EI} \right) dx_1 + \int_0^a \frac{\partial}{\partial F_A} \left(\frac{H_2^2}{2EI} \right) dx_2$$

$$0 = \int_0^a \frac{H_1}{EI} \frac{\partial H_1}{\partial F_A} dx_1 + \int_0^a \frac{H_2}{EI} \frac{\partial H_2}{\partial F_A} dx_2$$

$$0 = \frac{1}{EI} \int_0^a (-F_A x_1) (-x_1) dx_1 + \frac{1}{EI} \int_0^a [-F_A(x_2 + a) - F \cdot x_2] (-x_2 - a) dx_2$$

$$0 = \frac{1}{EI} F_A \frac{a^3}{3} + \frac{1}{EI} F_A \left(\frac{a^3}{3} + a^3 + a^3 \right) + \frac{1}{EI} F \left(\frac{a^3}{3} + \frac{a^3}{2} \right)$$

$$0 = F_A \frac{8}{3} + F \frac{5}{6}$$

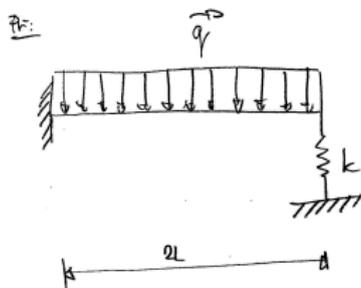
$$F_A = -F \frac{5}{8} \cdot \frac{3}{8} = -F \frac{15}{16}$$

Variational Methods: Examples

c) w_F evaluation using the Castigliano II theorem

$$\begin{aligned}w_F &= \frac{\partial U^*}{\partial F} = \int_0^a \frac{\partial}{\partial F} \left(\frac{M_1^2}{2EI} \right) dx_1 + \int_0^a \frac{\partial}{\partial F} \left(\frac{M_2^2}{2EI} \right) dx_2 = \int_0^a \frac{M_1}{EI} \frac{\partial M_1}{\partial F} dx_1 + \int_0^a \frac{M_2}{EI} \frac{\partial M_2}{\partial F} dx_2 \\&= \frac{1}{EI} \int_0^a \frac{5}{16} F \cdot x_1 \cdot \frac{5}{16} x_1 dx_1 + \frac{1}{EI} \int_0^a \left[\frac{5}{16} F (x_2 + a) - F x_2 \right] \left(\frac{5}{16} x_2 + \frac{5}{16} a - x_2 \right) dx_2 \\&= \frac{1}{EI} \frac{4}{96} F a^3\end{aligned}$$

Variational Methods: Examples



Consider a cantilever beam of length $2L$, bending stiffness EJ and supported by the spring of stiffness k . Derive the deflection of the beam in the end of it using the Euler-Bernoulli beam theory and using

- Unit-Dummy-Displacement method
- Unit-Dummy-load method
- Castigliano's theorem II
- Betti's reciprocity theorem

Variational Methods: Examples

a) We construct dummy disp. functions

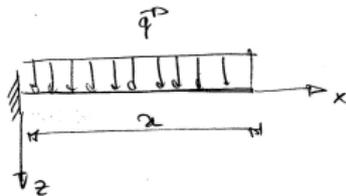
$$w^{iv} = \frac{1}{EI} q$$

$$w''' = \frac{1}{EI} qx + c_1$$

$$w'' = \frac{1}{EI} q \frac{x^2}{2} + c_1 x + c_2$$

$$w' = \frac{1}{EI} q \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3$$

$$w = \frac{1}{EI} q \frac{x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$$



$$\Rightarrow x=0 : w = w' = 0 \Rightarrow c_3 = c_4 = 0$$

$$x=l : w = w_2, w' = \varphi_2 \Rightarrow$$

$$\Rightarrow w_2 = \frac{1}{EI} q \frac{(2l)^4}{24} + c_1 \frac{(2l)^3}{6} + c_2 \frac{(2l)^2}{2}$$

$$\varphi_2 = \frac{1}{EI} q \frac{(2l)^3}{6} + c_1 \frac{(2l)^2}{2} + c_2 2l$$

$$c_1 = -\frac{2}{2l^3} w_2 + \frac{3}{2l^2} \varphi_2 - \frac{lq}{EI} \quad \leftarrow \downarrow$$



$$c_2 = \frac{3}{2l^2} w_2 - \frac{1}{l} \varphi_2 + \frac{l^2 q}{3EI}$$

$$w_0 = \left(\frac{3}{4l^2} x^2 - \frac{1}{4l^3} x^3 \right) w_2 + \left(-\frac{1}{2l} x^2 + \frac{1}{4l^2} x^3 \right) \varphi_2 + \left(\frac{l^2 x^2}{6} - \frac{lx^3}{6} + \frac{x^4}{24} \right) \frac{q}{EI}$$

$$w_0' = \left(\frac{3}{2l^2} x - \frac{3}{4l^3} x^2 \right) w_2 + \left(-\frac{1}{l} x + \frac{3}{4l^2} x^2 \right) \varphi_2 + \left(\frac{l^2 x}{3} - \frac{lx^2}{2} + \frac{x^3}{6} \right) \frac{q}{EI}$$

Variational Methods: Examples

Unit dummy displ. method:

$$\int_0^{2L} q \frac{\partial}{\partial w_{2L}} w_0 dx - k w_{2L} = \int_0^{2L} E I w_0'' \cdot \frac{\partial}{\partial w_{2L}} (w_0'') dx$$

$$\int_0^{2L} w_0 \frac{\partial}{\partial \varphi_{2L}} w_0' dx = \int_0^{2L} E I w_0'' \cdot \frac{\partial}{\partial \varphi_{2L}} (w_0'') dx$$

$$\int_0^{2L} q \cdot \phi_{w_{2L}} dx - k w_{2L} = \int_0^{2L} E I w_0'' \cdot \left(\frac{3}{2L^2} - \frac{3}{2L^3} x \right) dx$$

$$\int_0^{2L} 0 \cdot \phi_{\varphi_{2L}} dx = \int_0^{2L} E I w_0'' \cdot \left(\frac{1}{L} + \frac{3}{2L^2} x \right) dx$$

$$q \int_0^{2L} \left(\frac{3}{4L^2} x^2 - \frac{1}{4L^3} x^3 \right) dx - k w_{2L} = \int_0^{2L} E I \left[\left(\frac{3}{2L^2} - \frac{3}{2L^3} x \right) w_{2L} + \left(-\frac{1}{L} + \frac{3}{2L^2} x \right) \varphi_{2L} + \left(\frac{L^2}{3} - Lx + \frac{x^2}{2} \right) \frac{q}{E I} \right] \left(\frac{3}{2L^2} - \frac{3}{2L^3} x \right) dx \Rightarrow$$

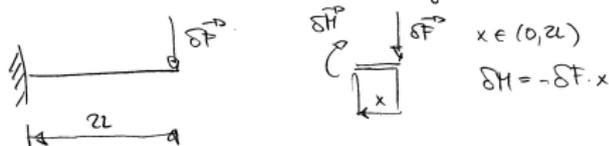


$$0 = \int_0^{2L} E I \left[\left(\frac{3}{2L^2} - \frac{3}{2L^3} x \right) w_{2L} + \left(-\frac{1}{L} + \frac{3}{2L^2} x \right) \varphi_{2L} + \left(\frac{L^2}{3} - Lx + \frac{x^2}{2} \right) \frac{q}{E I} \right] \left(-\frac{1}{L} + \frac{3}{2L^2} x \right) dx$$

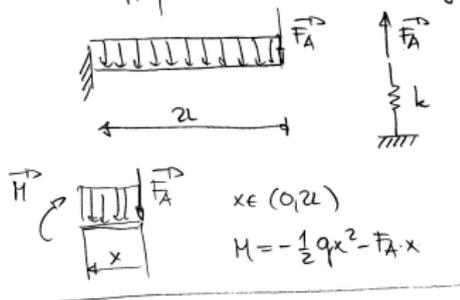
$$\Rightarrow \boxed{w_{2L} = \frac{6L^4 q}{3EI + 8L^3 k}}, \quad \varphi_{2L} = \frac{4L^3 q (-3EI + L^3 k)}{3EI (-3EI - 8L^3 k)}$$

Variational Methods: Examples

b) Let us introduce δH for unit dummy load



and apply it to the unit dummy load method



$$\delta F = 1, W_A = -\frac{1}{k} F_A$$

$$-\frac{1}{k} F_A = \int_0^{2L} \frac{1}{EI} \left(\frac{1}{2} q x^3 + F_A x^2 \right) dx$$

$$-\frac{1}{k} F_A = \frac{1}{EI} \left[\frac{1}{2} q \frac{x^4}{4} + F_A \frac{x^3}{3} \right]_0^{2L}$$

$$-\frac{1}{k} F_A = \frac{1}{EI} \left(\frac{1}{8} q (2L)^4 + \frac{1}{3} F_A (2L)^3 \right)$$

$$F_A \left(-\frac{1}{k} - \frac{1}{3EI} (2L)^3 \right) = \frac{1}{8EI} q (2L)^4$$

$$-F_A \left(8EI + \frac{8}{3} k (2L)^3 \right) = q (2L)^4$$

$$-F_A (24EI + 8k (2L)^3) = k 3q (2L)^4$$

$$F_A = -\frac{k 3q (2L)^4}{8(3EI + k(2L)^3)}$$

$$W_A \delta F = \int_0^{2L} \frac{1}{EI} H \delta H dx$$

$$W_A \delta F = \int_0^{2L} \frac{1}{EI} \left(-\frac{1}{2} q x^2 - F_A x \right) \cdot \delta F (-x) dx$$

$$W_A = -\frac{1}{k} F_A = \frac{3q(2L)^4}{8[3EI + k(2L)^3]}$$

Variational Methods: Examples

c) We find inner forces



$$T = qx - F_A$$

$$M = -\frac{1}{2}qx^2 + F_A x$$

for $x \in (0, 2L)$

Displacement condition at spring support + Castigliano's II theorem

$$\frac{\partial U}{\partial F_A} + \frac{1}{k} F_A = 0$$

$$\int_0^{2L} \frac{M}{EI} \frac{\partial M}{\partial F_A} dx + \frac{1}{k} F_A = 0$$

$$\frac{1}{EI} \int_0^{2L} \left(-\frac{1}{2}qx^2 + F_A x\right) \cdot x dx + \frac{1}{k} F_A = 0$$

$$\frac{1}{EI} \int_0^{2L} \left(-\frac{1}{2}qx^3 + F_A x^2\right) dx + \frac{1}{k} F_A = 0$$

$$\frac{1}{EI} \left[-\frac{1}{8}qx^4 + \frac{1}{3}F_A x^3 \right]_0^{2L} + \frac{1}{k} F_A = 0$$

$$\frac{1}{EI} \left(-\frac{1}{8}q(2L)^4 + \frac{1}{3}F_A(2L)^3\right) + \frac{1}{k} F_A = 0$$

$$F_A \left(\frac{1}{3EI} (2L)^3 + \frac{1}{k} \right) = \frac{1}{8EI} q(2L)^4$$

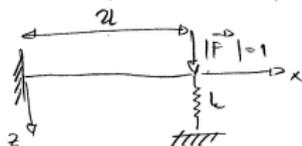
$$F_A = \frac{1}{8EI} \frac{q(2L)^4}{\frac{1}{3EI} (2L)^3 + \frac{1}{k}}$$

$$F_A = \frac{6L^4 q k}{3EI + 8L^3 k} \Rightarrow$$

$$\Rightarrow \boxed{w_A = \frac{6L^4 q}{3EI + 8L^3 k}}$$

Variational Methods: Examples

d) We find a fundamental solution of the problem w_0 :



$$w'''' = 0$$

$$w''' = c_1$$

$$w'' = c_1 x + c_2$$

$$w' = \frac{1}{2} c_1 x^2 + c_2 x + c_3$$

$$w = \frac{1}{6} c_1 x^3 + \frac{1}{2} c_2 x^2 + c_3 x + c_4$$

$$x=0: w = w' = 0 \Rightarrow c_3 = c_4 = 0$$

$$x=2L: w'' = 0$$

$$\left. \begin{aligned} w'' = 0 \\ w''' = -\frac{1}{EI} + \frac{k}{EI} w \end{aligned} \right\} \Rightarrow \begin{aligned} 0 &= c_1 2L + c_2 \Rightarrow c_2 = -c_1 2L \\ -\frac{1}{EI} + \frac{k}{EI} \left(\frac{1}{6} c_1 (2L)^3 + \frac{1}{2} c_2 (2L)^2 \right) &= c_1 \Rightarrow \end{aligned}$$

$$\Rightarrow -\frac{1}{EI} + \frac{k}{EI} c_1 \left(\frac{1}{6} (2L)^3 - \frac{1}{2} (2L)^2 \right) = c_1 \Rightarrow c_1 \left(1 - \frac{k}{EI} (2L)^2 \left(\frac{1}{6} - \frac{1}{2} \right) \right) = -\frac{1}{EI} \Rightarrow$$

$$\Rightarrow c_1 = \frac{-\frac{1}{EI}}{1 - \frac{k}{EI} (2L)^2 \frac{1-3}{6}} = \frac{-6}{6EI + 2k(2L)^2} = \frac{-3}{3EI + k(2L)^2} \Rightarrow c_2 = + \frac{3(2L)}{3EI + k(2L)^2}$$

$$w = \frac{1}{6} \frac{-3}{3EI + k(2L)^2} x^3 + \frac{1}{2} \frac{3(2L)}{3EI + k(2L)^2} x^2$$

$$w_0 = \frac{3}{3EI + k(2L)^2} \left(-\frac{1}{6} x^3 + \frac{1}{2} (2L) x^2 \right)$$

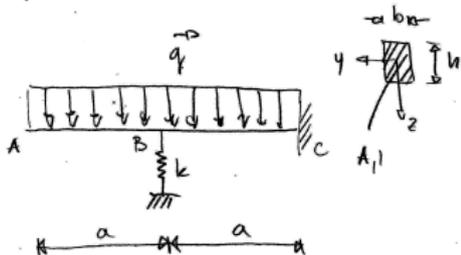
Variational Methods: Examples

and apply it to Betti's reciprocal theorem

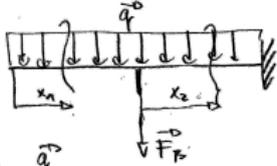
$$\begin{aligned} F w_{2L} &= \int_0^{2L} q \cdot w_0 dx = \int_0^{2L} \frac{3q}{3EI + k(2L)^3} \left[-\frac{1}{6}x^3 + \frac{1}{2}(2L)x^2 \right] dx \\ &= \frac{3q}{3EI + k(2L)^3} \left[-\frac{1}{24}(2L)^4 + \frac{1}{2}(2L) \frac{1}{3}(2L)^3 \right] = \\ &= \frac{3q(2L)^4}{3EI + k(2L)^3} \left(-\frac{1}{24} + \frac{1}{6} \right) = \frac{3q(2L)^4}{3EI + k(2L)^3} \frac{-1+4}{24} = \\ &= \frac{3q(2L)^4}{8(3EI + k(2L)^3)} \\ F=1 \Rightarrow & \boxed{w_{2L} = \frac{3q(2L)^4}{8[3EI + k(2L)^3]}} \end{aligned}$$

Variational Methods: Examples

Example 15: Derive the deflection of the beam at point B using the Castigliano's theorem II

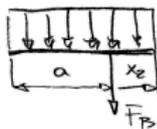


a) The internal moments



$$x_1 \in (0, a)$$

$$M_1 = -q \frac{x_1^2}{2}$$



$$x_2 \in (0, a)$$

$$M_2 = -q \frac{(x_2 + a)^2}{2} - F_B x_2$$

b) F_B evaluation using the Castigliano's theorem II

$$-F_B \cdot k^{-1} = \frac{\partial U^*}{\partial F_B}$$

$$-F_B \cdot k^{-1} = \int_0^a \frac{\partial}{\partial F_B} \left(\frac{M_1^2}{2EI} \right) dx_1 + \int_0^a \frac{\partial}{\partial F_B} \left(\frac{M_2^2}{2EI} \right) dx_2$$

$$-F_B \cdot k^{-1} = \frac{1}{EI} \left\{ \int_0^a M_1 \frac{\partial M_1}{\partial F_B} dx_1 + \int_0^a M_2 \frac{\partial M_2}{\partial F_B} dx_2 \right\}$$

$$-F_B \cdot k^{-1} = \frac{1}{EI} \left\{ \int_0^a (-q \frac{x_1^2}{2}) \cdot 0 dx_1 \right.$$

$$\left. + \int_0^a \left(-q \frac{(x_2 + a)^2}{2} - F_B x_2 \right) \cdot (-x_2) dx_2 \right\}$$

$$-F_B \cdot k^{-1} = \frac{1}{EI} \int_0^a \left[\frac{1}{2} q (x_2^3 + 2ax_2^2 + ax_2^2) + F_B x_2^2 \right] dx_2$$

Variational Methods: Examples

$$-F_B k^1 E l = \left[q \frac{x_2^4}{8} + 2aq \frac{x_2^3}{l} + a^2 q \frac{x_2^2}{4} + F_B \frac{x_2^3}{3} \right]_0^a$$

$$F_B \left(-k^1 E l - \frac{a^3}{3} \right) = q \left(\frac{a^4}{8} + 2 \frac{a^4}{l} + \frac{a^4}{4} \right)$$

$$F_B = - \frac{qa^4 \left(\frac{1}{8} + \frac{1}{l} + \frac{1}{4} \right)}{k^1 E l + \frac{a^3}{3}} \quad \frac{3+3+6}{24} = \frac{14}{24}$$

$$F_B = - \frac{14qa^4}{24k^1 E l + 8a^3} = - \frac{14qa^4 k}{8(3E l + a^3 k)}$$

c) w_B evaluation

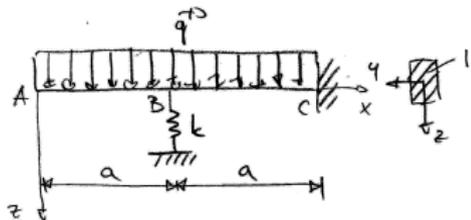
$$w_B = -F_B \cdot k^{-1} = \frac{14qa^4}{8(3E l + a^3 k)}$$

or

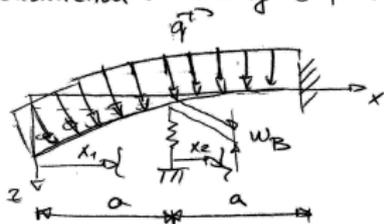
$$\begin{aligned} w_B &= \frac{\partial U^*}{\partial F_B} \Big|_{F_B} = \frac{1}{E l} \left[\frac{14}{24} qa^4 + F_B \cdot \frac{a^3}{3} \right] \Big|_{F_B} = \frac{14qa^4 k}{8(3E l + a^3 k)} \\ &= \frac{1}{E l} \left[\frac{14}{24} qa^4 - \frac{14qa^4}{24kE l + 8a^3} \frac{a^3}{3} \right] = \frac{1}{E l} \frac{\frac{14}{24} qa^4 (24kE l + 8a^3) - \frac{14}{3} qa^4}{24k^1 E l + 8a^3} = \underline{\underline{\frac{14qa^4}{8(3E l + a^3 k)}}} \end{aligned}$$

Variational Methods: Examples

Example 16: Derive the deflection of the beam at a point B using the unit-dummy displacement method.



a) Construction of dummy displacement functions



$$EI w_1'''' = q, \quad EI w_2'''' = q$$

$$EI w_1'''' = -qx_1 + C_1, \quad EI w_2'''' = qx_2 + D_1$$

$$EI w_1'' = \frac{1}{2}qx_1^2 + C_1x_1 + C_2$$

$$EI w_2'' = \frac{1}{2}qx_2^2 + D_1x_2 + D_2$$

$$EI w_1' = \frac{1}{6}qx_1^3 + \frac{1}{2}C_1x_1^2 + C_2x_1 + C_3$$

$$EI w_2' = \frac{1}{6}qx_2^3 + \frac{1}{2}D_1x_2^2 + D_2x_2 + D_3$$

$$EI w_1 = \frac{1}{24}qx_1^4 + \frac{1}{6}C_1x_1^3 + \frac{1}{2}C_2x_1^2 + C_3x_1 + C_4$$

$$EI w_2 = \frac{1}{24}qx_2^4 + \frac{1}{6}D_1x_2^3 + \frac{1}{2}D_2x_2^2 + D_3x_2 + D_4$$

Boundary and compatibility conditions are as follows

$$x_1 = 0: w_1'' = 0, w_1''' = 0$$

$$x_1 = a \wedge x_2 = 0: w_1 = w_2 = w_B$$

$$w_1'' = w_2'', w_1' = w_2'$$

$$x_2 = a: w_2 = w_2' = 0$$

and give the system of algebraic equations

$$0 = C_2$$

$$0 = C_4$$

$$EI w_B = \frac{1}{24}qa^4 + \frac{1}{6}Ca^3 + \frac{1}{2}C_2a^2 + C_3a + C_4$$

$$\frac{1}{24}qa^4 + Ca + C_2 = D_2$$

$$\frac{1}{6}qa^3 + \frac{1}{2}C_1a^2 + C_2a + C_3 = D_3$$

Variational Methods: Examples

$$EI w_B = D_4$$

$$0 = \frac{1}{24} q a^4 + \frac{1}{2} D_1 a^3 + \frac{1}{2} D_2 a^2 + D_3 a + D_4$$

$$0 = \frac{1}{6} q a^3 + \frac{1}{2} D_1 a^2 + D_2 a + D_3$$

with the solution

$$C_1 = C_2 = 0$$

$$C_3 = -\frac{13}{48} a^3 q - \frac{5}{2} \frac{EI}{a} w_B$$

$$C_4 = \frac{11}{48} a^4 q + \frac{5}{2} EI w_B$$

$$D_1 = -\frac{9}{8} a q + 3 \frac{EI}{a^3} w_B$$

$$D_2 = \frac{1}{2} a^2 q$$

$$D_3 = -\frac{5}{48} a^3 q - \frac{3}{2} \frac{EI}{a} w_B$$

$$D_4 = EI w_B$$

The elastic curves written using the dummy-disp.-function are

$$w_1 = \left(\frac{11}{48} a^4 - \frac{13}{48} a^3 x_1 + \frac{1}{24} x_1^4 \right) \frac{q}{EI} + \frac{1}{2} \left(5 - \frac{5}{a} x_1 \right) w_B$$

$$w_2 = \left(-\frac{5}{48} a^3 x_2 + \frac{1}{4} a^2 x_2^2 - \frac{5}{16} a x_2^3 + \frac{1}{24} x_2^4 \right) \frac{q}{EI} + \left(1 - \frac{3}{2a} x_2 + \frac{1}{2a^3} x_2^3 \right) w_B$$

b) The internal virtual work done due to the unit-displacement $w_B = 1$

$$\frac{1}{2} \int_0^a \delta [EI (w_1'')^2] \delta w_B = 1 \, dx_1 +$$

$$+ \frac{1}{2} \int_0^a \delta [EI (w_2'')^2] \delta w_B = 1 \, dx_2 =$$

$$= \frac{1}{2} EI \left\{ \int_0^a [2 w_1'' \cdot \delta w_1''] \delta w_B = 1 \, dx_1 + \right.$$

$$\left. + \int_0^a [2 w_2'' \cdot \delta w_2''] \delta w_B = 1 \, dx_2 \right\} =$$

$$= EI \left\{ \int_0^a \left[w_1' \cdot \frac{\partial w_1''}{\partial w_B} \right] dx_1 + \right.$$

$$\left. + \int_0^a \left[w_2' \cdot \frac{\partial w_2''}{\partial w_B} \right] dx_2 \right\} =$$

$$= EI \int_0^a \left[\left(\frac{1}{2} a^2 - \frac{9}{8} a x_2 + \frac{1}{2} x_2^2 \right) \frac{q}{EI} + \right.$$

$$\left. + \frac{5}{a^3} x_2 w_B \right] \frac{5}{a^3} x_2 dx_2$$

Variational Methods: Examples

c) The Unit - displacement - method:

$$\int_0^a q \cdot \delta w_1 \Big|_{\delta w_B=1} dx_1 + \int_0^a q \cdot \delta w_2 \Big|_{\delta w_B=1} dx_2 - k w_B = EI \left\{ \int_0^a w_1'' \frac{\partial w_1''}{\partial w_B} dx_1 + \int_0^a w_2'' \frac{\partial w_2''}{\partial w_B} dx_2 \right\}$$

$$\int_0^a q dx_1 + \int_0^a q \cdot \left(1 - \frac{3}{2a} x_2 + \frac{1}{2a^3} x_2^3 \right) dx_2 - k w_B = EI \int_0^a \left[\left(\frac{1}{2} a^2 - \frac{3}{8} a x_2 + \frac{1}{2} x_2^2 \right) \frac{q}{EI} + \frac{3}{a^3} x_2 w_B \right] \cdot \frac{3}{a^3} x_2 dx_2$$

It leads to the equation:

$$\frac{14aq}{8} + w_B \left(-\frac{3EI}{a^3} - k \right) = 0 \quad \Rightarrow \quad w_B = \frac{14a^3 q}{8(3EI + a^3 k)}$$

Variational Methods: Examples

```
#-----[           ]-----  
#-----[ Example 16 ]-----  
#-----[           ]-----  
  
#-----[ import library >sympy< ]-----  
#-----[ and setup best printing ]-----  
  
import sympy as sp  
sp.init_printing()  
  
#-----[ symbol declaration ]-----  
  
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')  
d1,d2,d3,d4=sp.symbols('d1 d2 d3 d4')  
E,I,k=sp.symbols('E I k')  
a,q,wB=sp.symbols('a q w_B')  
x1,x2=sp.symbols('x1 x2')
```

Variational Methods: Examples

#-----[algebraic equations for constants]-----

#-----[of the equations of the elastic curve]-----

$$\text{eqn1} = c2$$

$$\text{eqn2} = c1$$

$$\begin{aligned} \text{eqn3} = & -E \cdot I \cdot wB + \text{sp.Rational}(1, 24) \cdot q \cdot a^{**4} \setminus \\ & + \text{sp.Rational}(1, 6) \cdot c1 \cdot a^{**3} \setminus \\ & + \text{sp.Rational}(1, 2) \cdot c2 \cdot a^{**2} + c3 \cdot a + c4 \end{aligned}$$

$$\text{eqn4} = \text{sp.Rational}(1, 2) \cdot q \cdot a^{**2} + c1 \cdot a + c2 - d2$$

$$\begin{aligned} \text{eqn5} = & \text{sp.Rational}(1, 6) \cdot q \cdot a^{**3} \setminus \\ & + \text{sp.Rational}(1, 2) \cdot c1 \cdot a^{**2} + c2 \cdot a + c3 - d3 \end{aligned}$$

$$\text{eqn6} = E \cdot I \cdot wB - d4$$

$$\begin{aligned} \text{eqn7} = & \text{sp.Rational}(1, 24) \cdot q \cdot a^{**4} \setminus \\ & + \text{sp.Rational}(1, 6) \cdot d1 \cdot a^{**3} \setminus \\ & + \text{sp.Rational}(1, 2) \cdot d2 \cdot a^{**2} + d3 \cdot a + d4 \end{aligned}$$

$$\begin{aligned} \text{eqn8} = & \text{sp.Rational}(1, 6) \cdot q \cdot a^{**3} \setminus \\ & + \text{sp.Rational}(1, 2) \cdot d1 \cdot a^{**2} + d2 \cdot a + d3 \end{aligned}$$

Variational Methods: Examples

```
#-----[ ... and their solution ]-----
```

```
sol=sp.solve([eqn1,eqn2,eqn3,eqn4, \  
             eqn5,eqn6,eqn7,eqn8], \  
            [c1,c2,c3,c4,d1,d2,d3,d4])
```

```
print("c1=");sp.pprint(sol[c1])  
print("c2=");sp.pprint(sol[c2])  
print("c3=");sp.pprint(sol[c3])  
print("c4=");sp.pprint(sol[c4])  
print("d1=");sp.pprint(sol[d1])  
print("d2=");sp.pprint(sol[d2])  
print("d3=");sp.pprint(sol[d3])  
print("d4=");sp.pprint(sol[d4])
```

Variational Methods: Examples

```
#-----[ elastic curve for section I and II ]-----
```

```
w1=1/E/I*(sp.Rational(1,24)*q*x1**4 \
          +sp.Rational(1,6)*c1*x1**3 \
          +sp.Rational(1,2)*c2*x1**2+c3*x1+c4)
```

```
w2=1/E/I*(sp.Rational(1,24)*q*x2**4 \
          +sp.Rational(1,6)*d1*x2**3 \
          +sp.Rational(1,2)*d2*x2**2+d3*x2+d4)
```

```
w1_=w1.subs({c1:sol[c1],c2:sol[c2], \
             c3:sol[c3],c4:sol[c4]})
```

```
w2_=w2.subs({d1:sol[d1],d2:sol[d2], \
             d3:sol[d3],d4:sol[d4]})
```

Variational Methods: Examples

```
#-----[ elastic curves expressed using ]-----
#-----[ dummy-displacement-functions   ]-----

print("\nw1=")
sp.pprint(sp.expand(w1_).collect(wB).collect(q/E/I))
print("\nw2=")
sp.pprint(sp.expand(w2_).collect(wB).collect(q/E/I))
print("\nw1'=")
sp.pprint(sp.expand(w1_.diff(x1)) \
           .collect(wB).collect(q/E/I))
print("\nw2'=")
sp.pprint(sp.expand(w2_.diff(x2)) \
           .collect(wB).collect(q/E/I))
```

Variational Methods: Examples

```
print("\nw1''=")
sp.pprint(sp.expand(w1_.diff(x1,2)) \
           .collect(wB).collect(q/E/I))

print("\nw2''=")
sp.pprint(sp.expand(w2_.diff(x2,2)) \
           .collect(wB).collect(q/E/I))
```

Variational Methods: Examples

```
#-----[ system of algebraic equations for ]-----  
#-----[ Unit-displacement-method ...      ]-----  
  
eqn=sp.integrate(q*w1_.diff(wB),(x1,0,a)) \  
+sp.integrate(q*w2_.diff(wB),(x2,0,a))-k*wB \  
-E*I*sp.integrate( \  
    w1_.diff(x1,2)*w1_.diff(x1,x1,wB),(x1,0,a)) \  
-E*I*sp.integrate( \  
    w2_.diff(x2,2)*w2_.diff(x2,x2,wB),(x2,0,a))
```

Variational Methods: Examples

```
print("\nUnit-displacement-method algebraic equations:")
sp.pprint(sp.expand(eqn).collect(wB).collect(q/E/I))

#-----[ ... and solution ]-----

sol=sp.solve(eqn,wB)

print("wB=")
sp.pprint(sol[0])
```

Variational Methods: Examples

```
import matplotlib.pyplot as plt
import numpy as np

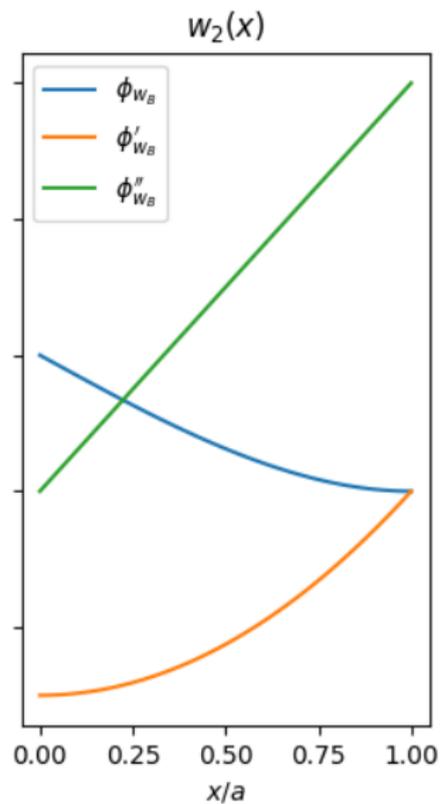
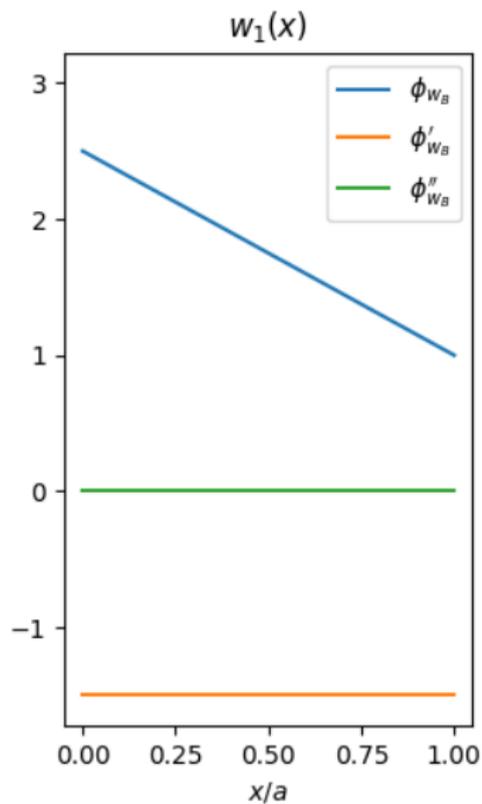
#-----[ make data ]-----

x=np.linspace(0,1,100);
y1=[w1_.diff(wB).subs({a:1,x1:ii}).evalf() for ii in x]
y2=[w2_.diff(wB).subs({a:1,x2:ii}).evalf() for ii in x]
y3=[w1_.diff(x1,wB).subs({a:1,x1:ii}).evalf() \
    for ii in x]
y4=[w2_.diff(x2,wB).subs({a:1,x2:ii}).evalf() \
    for ii in x]
y5=[w1_.diff(x1,x1,wB).subs({a:1,x1:ii}).evalf() \
    for ii in x]
y6=[w2_.diff(x2,x2,wB).subs({a:1,x2:ii}).evalf() \
    for ii in x]
```

Variational Methods: Examples

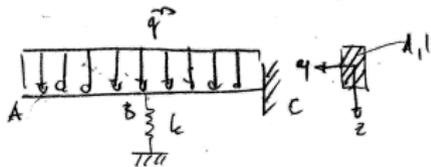
```
#-----[ plot ]-----  
  
fig, ([ax1,ax2])=plt.subplots(nrows=1, \  
                               ncols=2,sharey=True)  
ax1.set_title("$w_1(x)$")  
ax2.set_title("$w_2(x)$")  
ax1.plot(x,y1,label='$\phi_{w_B}$')  
ax1.plot(x,y3,label='$\phi_{w_B}^{\prime}$')  
ax1.plot(x,y5,label='$\phi_{w_B}^{\prime\prime}$')  
ax2.plot(x,y2,label='$\phi_{w_B}$')  
ax2.plot(x,y4,label='$\phi_{w_B}^{\prime}$')  
ax2.plot(x,y6,label='$\phi_{w_B}^{\prime\prime}$')  
ax1.set_xlabel("$x/a$")  
ax2.set_xlabel("$x/a$")  
ax1.legend()  
ax2.legend()  
plt.savefig('example16.png')
```

Variational Methods: Examples

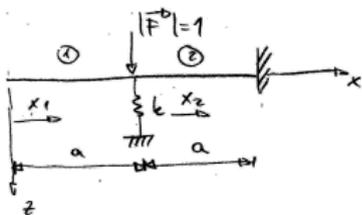


Variational Methods: Examples

Example 14: Let us have an indeterminate beam under the continuous loading q . It is considered to derive its deflection w_B



a) fundamental solution of the problem



$$w_1^{IV} = 0, \quad w_2^{IV} = 0$$

$$w_1''' = C_1, \quad w_2''' = D_1$$

$$w_1'' = C_1 x_1 + C_2, \quad w_2'' = D_1 x_2 + D_2$$

$$w_1' = \frac{1}{2} C_1 x_1^2 + C_2 x_1 + C_3, \quad w_2' = \frac{1}{2} D_1 x_2^2 + D_2 x_2 + D_3$$

$$w_1 = \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 + C_3 x_1 + C_4$$

$$w_2 = \frac{1}{6} D_1 x_2^3 + \frac{1}{2} D_2 x_2^2 + D_3 x_2 + D_4$$

boundary conditions:

$$x_1 = 0: w_1''' = w_1'' = 0$$

$$x_1 = a \wedge x_2 = 0:$$

$$w_1 = w_2, \quad w_1' = w_2', \quad w_1'' = w_2''$$

$$w_1''' = w_2''' - \frac{1}{EI} + \frac{k}{EI} w_1$$

$$x_2 = a: w_2 = w_2' = 0$$

and following system of algebraic system of equations:

$$0 = C_1$$

$$0 = C_2$$

$$\frac{1}{6} C_1 a^3 + \frac{1}{2} C_2 a^2 + C_3 a + C_4 = D_1$$

$$\frac{1}{2} C_1 a^2 + C_2 a + C_3 = D_3$$

Variational Methods: Examples

$$C_1 a + C_2 = D_2$$

$$\frac{C_1}{EI} = \frac{D_1}{EI} - \frac{1}{EI} + k \left(\frac{1}{2} C_1 a^3 + \frac{1}{2} C_2 a^2 + C_3 a + C_4 \right)$$

$$\frac{1}{2} D_1 a^3 + \frac{1}{2} D_2 a^2 + D_2 a + D_4 = 0$$

$$\frac{1}{2} D_1 a^2 + D_2 a + D_3 = 0$$

with the solution:

$$C_1 = C_2 = 0$$

$$C_3 = \frac{1}{3EI + a^3 k} \cdot \left(-\frac{3}{2} a^3 \right)$$

$$C_4 = \frac{1}{3EI + a^3 k} \cdot \frac{5}{2} a^3$$

$$D_1 = \frac{5}{3EI + a^3 k}$$

$$D_2 = 0$$

$$D_3 = \frac{1}{3EI + a^3 k} \cdot \left(-\frac{3}{2} a^2 \right)$$

$$D_4 = \frac{1}{3EI + a^3 k} \cdot a^2$$

$$\rightarrow \left[\begin{aligned} w_1 &= \frac{1}{3EI + a^3 k} \left(\frac{5}{2} a^3 - \frac{1}{2} a^2 x_1 \right) \\ w_2 &= \frac{1}{3EI + a^3 k} \left(a^3 - \frac{3}{2} a^2 x_2 + \frac{1}{2} x_2^3 \right) \end{aligned} \right]$$

b) The application of Betti's reciprocal theorem

$$F \cdot w_B = \int_0^a q w_1 dx_1 + \int_0^a q w_2 dx_2$$

$$|F| = 1 \Rightarrow$$

$$\Rightarrow w_B = \frac{q}{3EI + a^3 k} \left[\int_0^a \left(\frac{5}{2} a^3 - \frac{3}{2} a^2 x_1 \right) dx_1 + \right.$$

$$\left. + \int_0^a \left(a^3 - \frac{3}{2} a^2 x_2 + \frac{1}{2} x_2^3 \right) dx_2 \right]$$

$$= \frac{14 a^4 q}{8(3EI + a^3 k)}$$

Variational Methods: Examples

```
#-----[           ]-----  
#-----[ Example 17 ]-----  
#-----[           ]-----  
  
#-----[ import library >sympy< ]-----  
#-----[ and setup best printing ]-----  
  
import sympy as sp  
sp.init_printing()  
  
#-----[ symbol declaration ]-----  
  
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')  
d1,d2,d3,d4=sp.symbols('d1 d2 d3 d4')  
E,I,k=sp.symbols('E I k')  
a,q=sp.symbols('a q')  
x1,x2=sp.symbols('x1 x2')
```

Variational Methods: Examples

```
#-----[ algebraic equations for constants ]-----  
#-----[ of the equations of the elastic      ]-----  
#-----[ curve ...                          ]-----
```

$$\text{eqn1} = c1$$

$$\text{eqn2} = c2$$

$$\text{eqn3} = \text{sp.Rational}(1,6) * c1 * a^{**3} \backslash$$
$$+ \text{sp.Rational}(1,2) * c2 * a^{**2} + c3 * a + c4 - d4$$

$$\text{eqn4} = \text{sp.Rational}(1,2) * c1 * a^{**2} + c2 * a + c3 - d3$$

$$\text{eqn5} = c1 * a + c2 - d2$$

$$\text{eqn6} = c1 - d1 + 1 - E * I * k * (\text{sp.Rational}(1,6) * c1 * a^{**3} \backslash$$
$$+ \text{sp.Rational}(1,2) * c2 * a^{**2} + c3 * a + c4)$$

$$\text{eqn7} = \text{sp.Rational}(1,6) * d1 * a^{**3} \backslash$$
$$+ \text{sp.Rational}(1,2) * d2 * a^{**2} + d3 * a + d4$$

$$\text{eqn8} = \text{sp.Rational}(1,2) * d1 * a^{**2} + d2 * a + d3$$

Variational Methods: Examples

```
#-----[ algebraic equations for constants ]-----  
#-----[ ... and their solution ]-----
```

```
sol=sp.solve([eqn1,eqn2,eqn3,eqn4, \  
              eqn5,eqn6,eqn7,eqn8], \  
             [c1,c2,c3,c4,d1,d2,d3,d4])
```

```
print("c1=");sp.pprint(sol[c1])  
print("c2=");sp.pprint(sol[c2])  
print("c3=");sp.pprint(sol[c3])  
print("c4=");sp.pprint(sol[c4])  
print("d1=");sp.pprint(sol[d1])  
print("d2=");sp.pprint(sol[d2])  
print("d3=");sp.pprint(sol[d3])  
print("d4=");sp.pprint(sol[d4])
```

Variational Methods: Examples

```
#-----[ elastic curve for section I and II ]-----
```

```
w1=sp.Rational(1,6)*c1*x1**3 \  
    +sp.Rational(1,2)*c2*x1**2+c3*x1+c4
```

```
w2=sp.Rational(1,6)*d1*x2**3 \  
    +sp.Rational(1,2)*d2*x2**2+d3*x2+d4
```

```
w1_=w1.subs({c1:sol[c1],c2:sol[c2], \  
             c3:sol[c3],c4:sol[c4]})
```

```
w2_=w2.subs({d1:sol[d1],d2:sol[d2], \  
             d3:sol[d3],d4:sol[d4]})
```

Variational Methods: Examples

```
#-----[ elastic curves expressed using ]-----  
#-----[ dummy-displacement-functions   ]-----  
  
print("\nw1=")  
sp.pprint(sp.expand(w1_).collect(wB))  
print("\nw2=")  
sp.pprint(sp.expand(w2_).collect(wB))  
  
#-----[ Betti's theorem ]-----  
  
wB=sp.integrate(q*w1_,(x1,0,a)) \  
    +sp.integrate(q*w2_,(x2,0,a))  
  
print("wB=")  
sp.pprint(sp.simplify(wB))
```

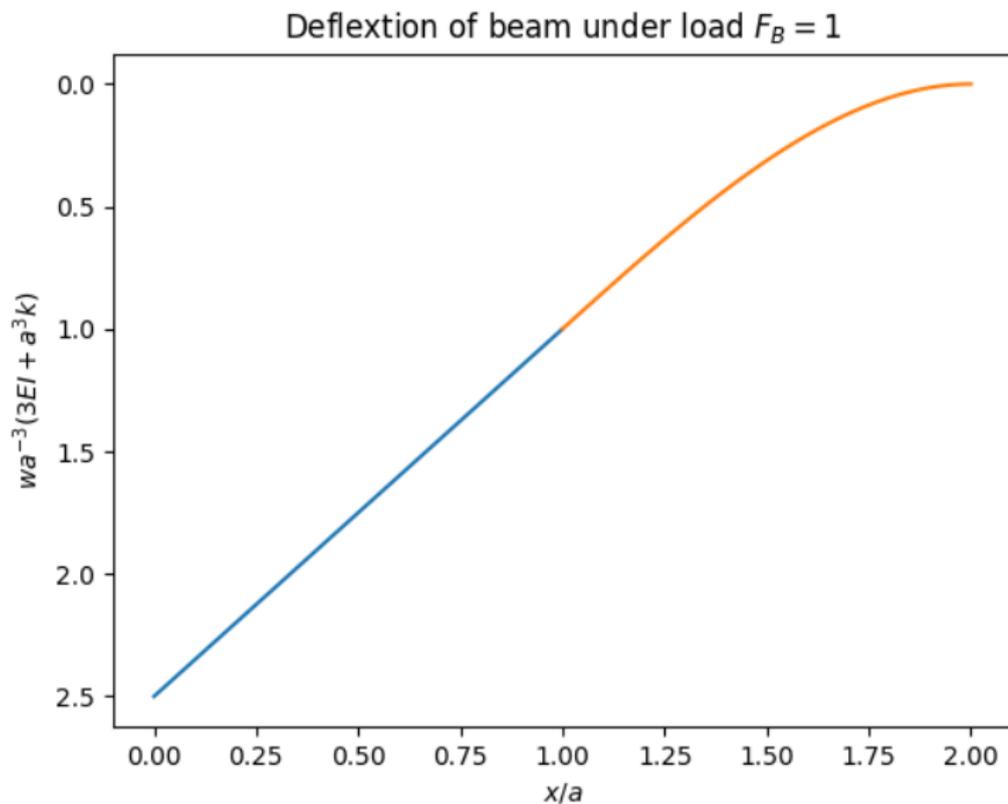
Variational Methods: Examples

```
#-----[ Plot of fundamental solution ]-----  
  
import matplotlib.pyplot as plt  
import numpy as np  
  
#-----[ make data ]-----  
  
x=np.linspace(0,1,100);  
xx=np.linspace(1,2,100);  
  
y1=[w1_.subs({a:1,k:-2,I:1,E:1,x1:ii}).evalf() \  
    for ii in x]  
y2=[w2_.subs({a:1,k:-2,I:1,E:1,x2:ii}).evalf() \  
    for ii in x]
```

Variational Methods: Examples

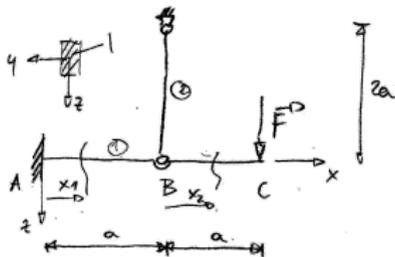
```
#-----[ plot ]-----  
  
fig,ax=plt.subplots()  
  
ax.set_title("Deflection of beam under load  $F_B=1$ ")  
ax.invert_yaxis()  
ax.plot(x,y1)  
ax.plot(xx,y2)  
  
ax.set_xlabel(" $x/a$ ")  
ax.set_ylabel(" $wa^{-3}(3EI+a^3k)$ ")  
  
plt.savefig('example17.png')
```

Variational Methods: Examples

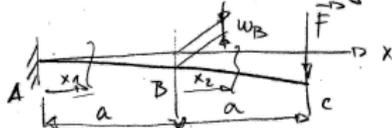


Variational Methods: Examples

Example 18: Consider a beam and a bar connected at point B. Derive the deflection of the beam at point B using unit-dummy-displacement method.



a) A construction of dummy-displ. functions



$$w_1^{IV} = 0, w_2^{IV} = 0$$

$$w_1''' = c_1, w_2''' = D_1$$

$$w_1'' = c_1 x_1 + c_2, w_2'' = D_1 x_2 + D_2$$

$$w_1' = \frac{1}{2} c_1 x_1^2 + c_2 x_1 + c_3, w_2' = \frac{1}{2} D_1 x_2^2 + D_2 x_2 + D_3$$

$$w_1 = \frac{1}{6} c_1 x_1^3 + \frac{1}{2} c_2 x_1^2 + c_3 x_1 + c_4$$

$$w_2 = \frac{1}{6} D_1 x_2^3 + \frac{1}{2} D_2 x_2^2 + D_3 x_2 + D_4$$

with boundary and compatibility conditions

$$x_1 = 0: w_1 = w_1' = 0$$

$$x_1 = a \wedge x_2 = 0: w_1 = w_2 = w_B$$

$$w_1' = w_2', w_1'' = w_2''$$

$$x_2 = a: w_2'' = 0, w_2''' = -\frac{F}{EI}$$

and leading to the equations

$$c_1 = 0$$

$$c_3 = 0$$

$$\frac{1}{6} c_1 a^3 + \frac{1}{2} c_2 a^2 + c_3 a + c_4 = w_B$$

$$D_4 = w_B$$

$$\frac{1}{2} c_1 a^2 + c_2 a + c_3 = D_3$$

$$c_1 a + c_2 = D_2$$

$$D_1 a + D_2 = 0$$

$$D_1 = -\frac{F}{EI}$$

Variational Methods: Examples

Their solution is

$$c_1 = -\frac{3}{a^2} w_B - \frac{5}{2EI} F$$

$$c_2 = \frac{3}{a^2} w_B + \frac{a}{2EI} F$$

$$c_3 = c_4 = 0$$

$$d_1 = \frac{1}{EI} F$$

$$d_2 = -\frac{a}{EI} F$$

$$d_3 = \frac{3}{2a} w_B - \frac{a^2}{4EI} F$$

$$d_4 = w_B$$

The dummy-displ. functions are

$$w_1 = \left(\frac{3}{2a^2} x_1^2 - \frac{1}{2a^2} x_1^3 \right) w_B + \frac{1}{4EI} (a+x_1) x_1^2 F$$

$$w_2 = \left(1 + \frac{3}{2a} x_2 \right) w_B + \frac{1}{2EI} \left(-\frac{a^2}{2} - a x_2 + \frac{1}{3} x_2^2 \right) x_2 F$$

$$u_3 = \frac{x_3}{2a} w_B$$

b) The Unit-dummy-displ. method

$$F \delta w_2 \Big|_{x_2=a} \Big|_{\delta w_B=1} = \int_0^a \frac{1}{2} EI \delta (w_1'')^2 \Big|_{\delta w_B=1} dx_1 + \int_0^a EI \delta (w_2'')^2 \Big|_{\delta w_B=1} dx_2 + ES \int_0^{2a} \delta (u_3')^2 dx_3$$

$$F \cdot \frac{\partial w_2}{\partial w_B} \Big|_{x_2=a} = EI \left\{ \int_0^a w_1'' \delta w_1'' \Big|_{\delta w_B=1} dx_1 + \int_0^a w_2'' \delta w_2'' \Big|_{\delta w_B=1} dx_2 \right\} + ES \int_0^{2a} u_3' \delta u_3' \Big|_{\delta w_B=1} dx_3$$

$$F \cdot \left(1 + \frac{3}{2a} x_2 \right) \Big|_{x_2=a} = EI \left\{ \int_0^a w_1'' \frac{\partial w_1''}{\partial w_B} dx_1 + \int_0^a w_2'' \frac{\partial w_2''}{\partial w_B} dx_2 \right\} + ES \int_0^{2a} u_3' \frac{\partial u_3'}{\partial w_B} dx_3$$

↳ →

Variational Methods: Examples

$$\rightarrow \frac{\Sigma}{2} F = EI \left\{ \int_0^a \left[\left(\frac{z}{a^2} - \frac{3x_1}{a^3} \right) w_B + \left(\frac{a}{2EI} - \frac{5x_1}{2EI} \right) F \right] \cdot \left(\frac{z}{a^2} - \frac{3x_1}{a^3} \right) dx_1 + \int_0^a \left(-\frac{a}{EI} + \frac{x_2}{EI} \right) F \cdot \Phi \downarrow x_2 \right\} + ES \int_0^{2a} \frac{1}{2a} w_B \cdot \frac{1}{2a} dx_3$$

$$\frac{\Sigma}{2} F = EI \frac{3}{a^3} w_B + ES \frac{1}{2a} w_B \Rightarrow \boxed{w_B = \frac{5Fa^3}{E(6I + 5a^2E)}}$$

Variational Methods: Examples

```
#-----[           ]-----  
#-----[ Example 18 ]-----  
#-----[           ]-----  
  
#-----[ import library >sympy< ]-----  
#-----[ and setup best printing ]-----  
  
import sympy as sp  
sp.init_printing()  
  
#-----[ symbol declaration ]-----  
  
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')  
d1,d2,d3,d4=sp.symbols('d1 d2 d3 d4')  
E,I,S,F=sp.symbols('E I S F')  
a,wB=sp.symbols('a w_B')  
x1,x2,x3=sp.symbols('x1 x2 x3')
```

Variational Methods: Examples

```
#-----[ algebraic equations for constants ]-----  
#-----[ of the equations of the elastic   ]-----  
#-----[ curve ...                         ]-----
```

$$\text{eqn1} = c4$$

$$\text{eqn2} = c3$$

$$\text{eqn3} = -wB + \text{sp.Rational}(1,6) * c1 * a^{**3} \backslash$$
$$\quad + \text{sp.Rational}(1,2) * c2 * a^{**2} + c3 * a + c4$$

$$\text{eqn4} = -wB + d4$$

$$\text{eqn5} = \text{sp.Rational}(1,2) * c1 * a^{**2} + c2 * a + c3 - d3$$

$$\text{eqn6} = c1 * a + c2 - d2$$

$$\text{eqn7} = d1 * a + d2$$

$$\text{eqn8} = d1 - F/E/I$$

Variational Methods: Examples

```
#-----[ ... and their solution ]-----
```

```
sol=sp.solve([eqn1,eqn2,eqn3,eqn4, \  
             eqn5,eqn6,eqn7,eqn8], \  
            [c1,c2,c3,c4,d1,d2,d3,d4])
```

```
print("c1=");sp.pprint(sol[c1])  
print("c2=");sp.pprint(sol[c2])  
print("c3=");sp.pprint(sol[c3])  
print("c4=");sp.pprint(sol[c4])  
print("d1=");sp.pprint(sol[d1])  
print("d2=");sp.pprint(sol[d2])  
print("d3=");sp.pprint(sol[d3])  
print("d4=");sp.pprint(sol[d4])
```

Variational Methods: Examples

```
#-----[ elastic curve for section I and II ]-----
```

```
w1=sp.Rational(1,6)*c1*x1**3 \
    +sp.Rational(1,2)*c2*x1**2+c3*x1+c4
w2=sp.Rational(1,6)*d1*x2**3 \
    +sp.Rational(1,2)*d2*x2**2+d3*x2+d4
u3=wB*x3/2/a
```

```
w1_=w1.subs({c1:sol[c1],c2:sol[c2], \
             c3:sol[c3],c4:sol[c4]})
w2_=w2.subs({d1:sol[d1],d2:sol[d2], \
             d3:sol[d3],d4:sol[d4]})
u3_=u3
```

Variational Methods: Examples

```
#-----[ elastic curves expressed using ]-----
#-----[ dummy-displacement-functions   ]-----

print("\nw1=")
sp.pprint(sp.expand(w1_).collect(wB).collect(F))
print("\nw2=")
sp.pprint(sp.expand(w2_).collect(wB).collect(F))
print("\nu3=")
sp.pprint(sp.expand(u3_).collect(wB))
print("\nw1'=")
sp.pprint(sp.expand(w1_.diff(x1)).collect(wB) \
           .collect(F))
print("\nw2'=")
sp.pprint(sp.expand(w1_.diff(x1)).collect(wB) \
           .collect(F))
print("\nu3'=")
```

Variational Methods: Examples

```
sp.pprint(sp.expand(u3_.diff(x3)).collect(wB))
print("\nw1''=")
sp.pprint(sp.expand(w1_.diff(x1,2)).collect(wB) \
          .collect(F))
print("\nw2''=")
sp.pprint(sp.expand(w2_.diff(x2,2)).collect(wB) \
          .collect(F))
print("\nu3''=")
sp.pprint(sp.expand(u3_.diff(x3,2)) \
          .collect(wB))
```

Variational Methods: Examples

```
#-----[ system of algebraic equations for ]-----  
#-----[ Unit-displacement-method ...      ]-----  
  
eqn=-F*w2_.diff(wB).subs(x2,a) \  
+E*I*sp.integrate(w1_.diff(x1,2)*w1_.diff(x1,x1,wB), \  
                  (x1,0,a)) \  
+E*I*sp.integrate(w2_.diff(x2,2)*w2_.diff(x2,x2,wB), \  
                  (x2,0,a)) \  
+E*S*sp.integrate(u3_.diff(x3)*u3_.diff(x3,wB), \  
                  (x3,0,2*a))  
  
print("\nUnit-displcement-method algebraic equations:")  
sp.pprint(sp.expand(eqn))
```

Variational Methods: Examples

```
#-----[ ... and solution ]-----
```

```
sol=sp.solve(eqn,wB)
```

```
print("wB=")
```

```
sp.pprint(sol[0])
```

Variational Methods: Examples

```
import matplotlib.pyplot as plt
import numpy as np

#-----[ make data ]-----

x=np.linspace(0,1,100);
xx=np.linspace(0,2,100);
y1=[w1_.diff(wB).subs({a:1,x1:ii}).evalf() for ii in x]
y2=[w2_.diff(wB).subs({a:1,x2:ii}).evalf() for ii in x]
y3=[u3_.diff(wB).subs({a:1,x3:ii}).evalf() for ii in xx]
y4=[w1_.diff(x1,wB).subs({a:1,x1:ii}).evalf() \
    for ii in x]
y5=[w2_.diff(x2,wB).subs({a:1,x2:ii}).evalf() \
    for ii in x]
y6=[u3_.diff(x3,wB).subs({a:1,x3:ii}).evalf() \
    for ii in xx]
```

Variational Methods: Examples

```
y7=[w1_.diff(x1,x1,wB).subs({a:1,x1:ii}).evalf() \  
    for ii in x]  
y8=[w2_.diff(x2,x2,wB).subs({a:1,x2:ii}).evalf() \  
    for ii in x]  
y9=[u3_.diff(x3,x3,wB).subs({a:1,x3:ii}).evalf() \  
    for ii in xx]
```

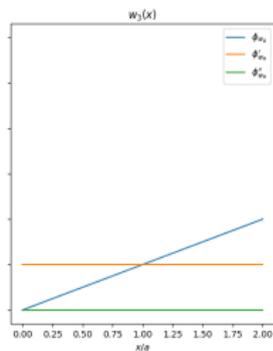
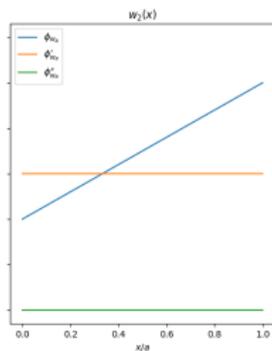
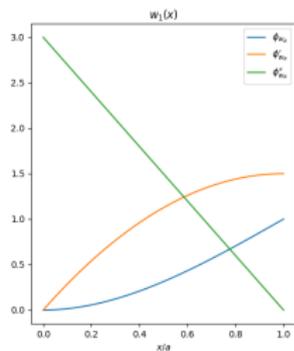
Variational Methods: Examples

```
#-----[ plot ]-----  
  
fig,([ax1,ax2,ax3])=plt.subplots(nrows=1,ncols=3, \  
    sharey=True,figsize=(18,6))  
  
ax1.set_title("$w_1(x)$")  
ax2.set_title("$w_2(x)$")  
ax3.set_title("$w_3(x)$")  
  
ax1.plot(x,y1,label='$\phi_{w_B}$')  
ax1.plot(x,y4,label='$\phi_{w_B}^\prime$')  
ax1.plot(x,y7,label='$\phi_{w_B}^{\prime\prime}$')  
ax2.plot(x,y2,label='$\phi_{w_B}$')  
ax2.plot(x,y5,label='$\phi_{w_B}^\prime$')  
ax2.plot(x,y8,label='$\phi_{w_B}^{\prime\prime}$')
```

Variational Methods: Examples

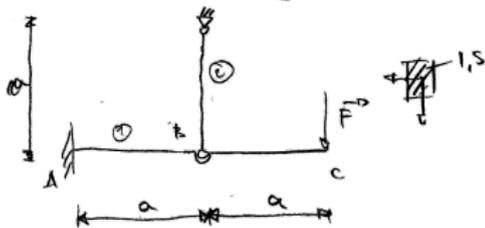
```
ax3.plot(xx,y3,label='$\phi_{w_B}$')  
ax3.plot(xx,y6,label='$\phi_{w_B}^{\prime}$')  
ax3.plot(xx,y9,label='$\phi_{w_B}^{\prime\prime}$')  
  
ax1.set_xlabel("$x/a$")  
ax2.set_xlabel("$x/a$")  
ax3.set_xlabel("$x/a$")  
  
ax1.legend()  
ax2.legend()  
ax3.legend()  
  
plt.savefig('example18.png')
```

Variational Methods: Examples

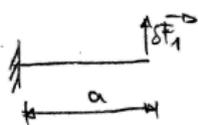


Variational Methods: Examples

Example 19: Consider a beam and bar connected at point B. Derive the deflection and elongation of the beam and bar, respectively, at point B using the Unit-dummy-load method.



a) $\delta M_{1,2}$ and δN_3 for unit-dummy-load

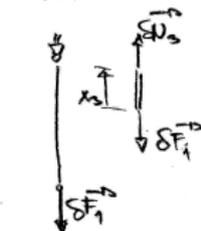
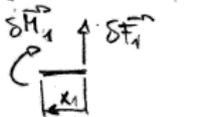


$$x_1 \in (0, a)$$

$$\delta M_1 = \delta F_1 \cdot x_1$$

$$x_2 \in (0, a)$$

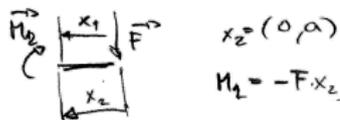
$$\delta M_2 = 0$$



$$x_3 \in (0, 1.5)$$

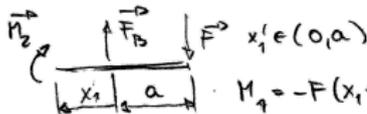
$$\delta N_3 = \delta F_1$$

b) internal forces and moments



$$x_2 \in (0, a)$$

$$M_2 = -F \cdot x_2$$



$$x_1' \in (0, a)$$

$$M_1 = -F(x_1 + a) +$$

$$+ F_B \cdot x_1$$



$$x_3 \in (0, 1.5)$$

$$N_3 = F_B$$

Variational Methods: Examples

c) Unit-dummy-load method

$$\delta F_1 = 1, \quad w_{1B} = u_{3B}$$

$$-w_{1B} \delta F_1 + u_{3B} \delta F_1 = \int_0^a \frac{1}{EI} M_1 \delta M_1 dx_1 + \int_0^a \frac{1}{EI} M_2 \delta M_2 dx_2 + \int_0^{2a} \frac{1}{ES} N_3 \delta N_3 dx_3$$

$$(-w_{1B} + u_{3B}) \delta F_1 = \frac{1}{EI} \int_0^a [-F(x_1+a) + F_B x_1] \cdot \delta F_1 x_1 dx_1 + \frac{1}{EI} \int_0^a (-F x_2) \cdot 0 dx_2 + \frac{1}{ES} \int_0^{2a} F_B \cdot \delta F_1 dx_3$$

$$0 = \frac{1}{EI} \int_0^a [-F(x_1^2 + ax_1) + F_B x_1^2] dx_1 + \frac{1}{ES} \int_0^{2a} F_B dx_3$$

$$0 = \frac{1}{EI} \left[-F \left(\frac{x_1^3}{3} + a \frac{x_1^2}{2} \right) + F_B \frac{x_1^3}{3} \right]_0^a + \frac{1}{ES} [F_B x_3]_0^{2a}$$

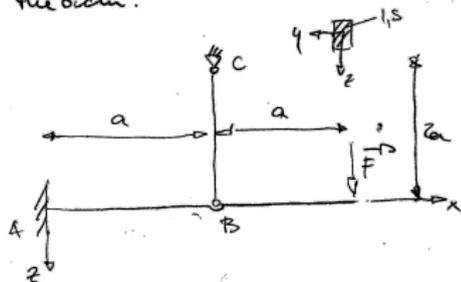
$$0 = \frac{1}{EI} \left[-Fa^3 \left(\frac{1}{3} + \frac{1}{2} \right) + F_B a^3 \frac{1}{3} \right] + \frac{1}{ES} F_B 2a$$

$$0 = \frac{5Fa^2}{EI} + F_B \left(\frac{a^2}{3I} + \frac{2}{ES} \right) \Rightarrow F_B = \frac{5Fa^2 \cdot 3ES}{2ES(3a^2 + 6I)} = \frac{5F5a^2}{2(3a^2 + 6I)}$$

$$u_B = w_B = \frac{N_3 \cdot 2a}{ES} = \frac{F_B \cdot 2a}{ES} = \frac{5F5a^2 \cdot 2a}{2ES(3a^2 + 6I)} = \frac{5Fa^3}{ES(3a^2 + 6I)}$$

Variational Methods: Examples

Example 20: Derive the deflection of the beam at point B using the Betti's reciprocity theorem.



a) Derivation of fundamental solution

$$w_1^{IV} = 0, w_2^{IV} = 0$$

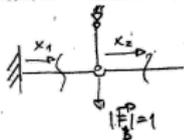
$$w_1''' = C_1, w_2''' = D_1$$

$$w_1'' = C_1 x_1 + C_2, w_2'' = D_1 x_2 + D_2$$

$$w_1' = \frac{1}{2} C_1 x_1^2 + C_2 x_1 + C_3, w_2' = \frac{1}{2} D_1 x_2^2 + D_2 x_2 + D_3$$

$$w_1 = \frac{1}{6} C_1 x_1^3 + \frac{1}{2} C_2 x_1^2 + C_3 x_1 + C_4$$

$$w_2 = \frac{1}{6} D_1 x_2^3 + \frac{1}{2} D_2 x_2^2 + D_3 x_2 + D_4$$



with boundary and compatibility conditions:

$$x_1 = 0: w_1 = w_1' = 0$$

$$x_1 = a, x_2 = 0: w_1 = w_2, w_1' = w_2', w_1'' = w_2''$$

$$w_1''' = w_2''' - \frac{1}{EI} + \frac{3}{12a} w_2$$

$$x_2 = a: w_2'' = 0, w_2''' = 0$$

which lead to the algebraic system of equations

$$C_3 = 0$$

$$C_4 = 0$$

$$\frac{1}{6} C_1 a^3 + \frac{1}{2} C_2 a^2 + C_3 a + C_4 = D_4$$

$$\frac{1}{2} C_1 a^2 + C_2 a + C_3 = D_3$$

$$C_1 a + C_2 = D_2$$

$$C_1 = D_1 - \frac{1}{EI} + \frac{3}{12a} D_3$$

$$D_1 a + D_2 = 0$$

$$D_1 = 0$$

Variational Methods: Examples

Its solution is

$$c_1 = \frac{-C}{E(6l + 5a^2)}$$

$$c_2 = \frac{6a}{E(6l + 5a^2)}$$

$$c_3 = c_4 = D_1 = D_2 = 0$$

$$D_3 = \frac{3a^2}{E(6l + 5a^2)}$$

$$D_4 = \frac{2a^3}{E(6l + 5a^2)}$$

The fundamental solution is

$$w_1 = \frac{1}{6lE + 5a^2E} (3ax_1^2 - x_1^3)$$

$$w_2 = \frac{a^2}{6lE + 5a^2E} (2a + 3x_2)$$

b) Betti's reciprocity theorem

$$F_B \cdot w_B = F \cdot w_F \Big|_{x_2=a}$$

$$w_B = F \cdot \frac{a^2}{6lE + 5a^2E} (2a + 3a)$$

$$w_B = \frac{5a^3 F}{E(6l + 5a^2)}$$

Variational Methods: Examples

```
#-----[           ]-----  
#-----[ Example 20 ]-----  
#-----[           ]-----  
  
#-----[ import library >sympy< ]-----  
#-----[ and setup best printing ]-----  
  
import sympy as sp  
sp.init_printing()  
  
#-----[ symbol declaration ]-----  
  
c1,c2,c3,c4=sp.symbols('c1 c2 c3 c4')  
d1,d2,d3,d4=sp.symbols('d1 d2 d3 d4')  
E,I=sp.symbols('E I')  
a,F=sp.symbols('a F')  
x1,x2=sp.symbols('x1 x2')
```

Variational Methods: Examples

```
#-----[ algebraic equations for constants ]-----  
#-----[ of the equations of the elastic      ]-----  
#-----[ curve ...                          ]-----
```

$$\text{eqn1} = c3$$

$$\text{eqn2} = c4$$

$$\text{eqn3} = \text{sp.Rational}(1,6) * c1 * a^{**3} + \text{sp.Rational}(1,2) * c2 * a^{**2} \ \backslash$$
$$+ c3 * a + c4 - d4$$

$$\text{eqn4} = \text{sp.Rational}(1,2) * c1 * a^{**2} + c2 * a + c3 - d3$$

$$\text{eqn5} = c1 * a + c2 - d2$$

$$\text{eqn6} = c1 - d1 + 1/E/I - S/I/2/a * (\text{sp.Rational}(1,6) * c1 * a^{**3} \ \backslash$$
$$+ \text{sp.Rational}(1,2) * c2 * a^{**2} + c3 * a + c4)$$

$$\text{eqn7} = d1 * a + d2$$

$$\text{eqn8} = d1$$

Variational Methods: Examples

```
#-----[ ... and their solution ]-----
```

```
sol=sp.solve([eqn1,eqn2,eqn3,eqn4, \  
             eqn5,eqn6,eqn7,eqn8], \  
            [c1,c2,c3,c4,d1,d2,d3,d4])
```

```
print("c1=");sp.pprint(sol[c1])  
print("c2=");sp.pprint(sol[c2])  
print("c3=");sp.pprint(sol[c3])  
print("c4=");sp.pprint(sol[c4])  
print("d1=");sp.pprint(sol[d1])  
print("d2=");sp.pprint(sol[d2])  
print("d3=");sp.pprint(sol[d3])  
print("d4=");sp.pprint(sol[d4])
```

Variational Methods: Examples

```
#-----[ elastic curve for section I and II ]-----
```

```
w1=sp.Rational(1,6)*c1*x1**3 \
    +sp.Rational(1,2)*c2*x1**2+c3*x1+c4
```

```
w2=sp.Rational(1,6)*d1*x2**3 \
    +sp.Rational(1,2)*d2*x2**2+d3*x2+d4
```

```
w1_=w1.subs({c1:sol[c1],c2:sol[c2], \
             c3:sol[c3],c4:sol[c4]})
```

```
w2_=w2.subs({d1:sol[d1],d2:sol[d2], \
             d3:sol[d3],d4:sol[d4]})
```

Variational Methods: Examples

```
#-----[ elastic curves expressed using ]-----  
#-----[ dummy-displacement-functions   ]-----  
  
print("\nw1=")  
sp.pprint(sp.expand(w1_).collect(wB).collect(F))  
print("\nw2=")  
sp.pprint(sp.expand(w2_).collect(wB).collect(F))  
  
#-----[ Betti's theorem ]-----  
  
wB=F*w2_.subs(x2,a)  
  
print("wB=")  
sp.pprint(sp.simplify(wB))
```

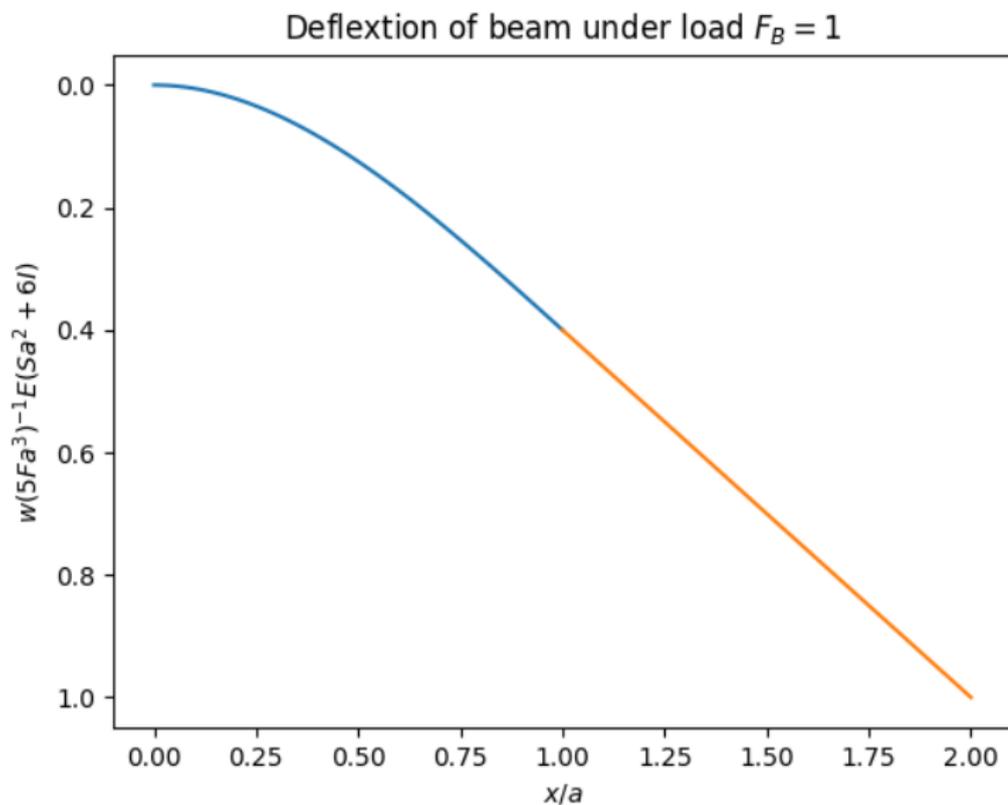
Variational Methods: Examples

```
#-----[ Plot of fundamental solution ]-----  
  
import matplotlib.pyplot as plt  
import numpy as np  
  
#-----[ make data ]-----  
  
x=np.linspace(0,1,100);  
xx=np.linspace(1,2,100);  
  
y1=[w1_.subs({a:1,S:-1,I:1,E:1,x1:ii}).evalf() \  
    for ii in x]  
y2=[w2_.subs({a:1,S:-1,I:1,E:1,x2:ii}).evalf() \  
    for ii in x]
```

Variational Methods: Examples

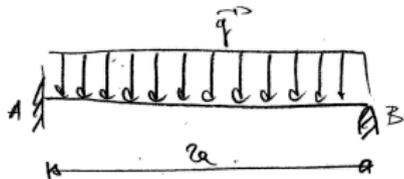
```
#-----[ plot ]-----  
  
fig,ax=plt.subplots()  
  
ax.set_title("Deflection of beam under load  $F_B=1$ ")  
ax.invert_yaxis()  
  
ax.plot(x,y1)  
ax.plot(xx,y2)  
  
ax.set_xlabel(" $x/a$ ")  
ax.set_ylabel(" $w(5Fa^3)^{-1}E(Sa^2+6I)$ ")  
  
plt.savefig('example20.png')
```

Variational Methods: Examples

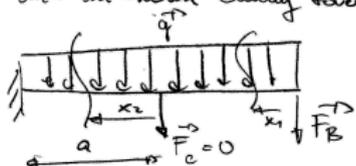


Variational Methods: Examples

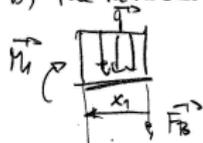
Example 22: Consider a supported cantilever in the Figure. Let derive the deflection of the cantilever at the center using the Castigliano's II theorem



- a) Introducing the dummy zero valued force at the center of the cantilever and unknown dummy force at the end B,

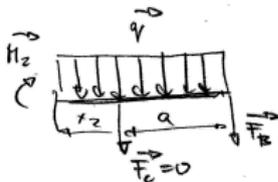


- b) The internal resulting moments



$$x_1 \in (0, a)$$

$$H_1 = -q \frac{x_1^2}{2} - F_B \cdot x_1$$



$$x_2 \in (0, a)$$

$$H_2 = -q \frac{(x_2+a)^2}{2} -$$

$$-F_B(x_2+a) -$$

$$-F_c \cdot x_2$$

- c) The evaluation of F_B using the second Castigliano's theorem

$$W_B = 0 = \frac{\partial U^*}{\partial F_B}$$

$$0 = \frac{\partial}{\partial F_B} \left\{ \int_0^a \frac{H_1^2}{2EI} dx_1 + \int_0^a \frac{H_2^2}{2EI} dx_2 \right\}$$

$$0 = \int_0^a \frac{H_1}{EI} \frac{\partial H_1}{\partial F_B} dx_1 + \int_0^a \frac{H_2}{EI} \frac{\partial H_2}{\partial F_B} dx_2$$

Variational Methods: Examples

$$0 = \int_0^a \left(-q \frac{x_1^2}{2} - F_B x_1 \right) (-x_1) dx_1 + \int_0^a \left(-q \frac{(x_2+a)^2}{2} - F_B (x_2+a) - F_C x_2 \right) (-x_2-a) dx_2$$

$$0 = \left[\frac{F_B x_1^3}{3} + \frac{q x_1^4}{8} \right]_0^a + \left[\frac{q x_2^4}{8} + x_2^3 \left(\frac{F_B}{3} + \frac{F_C}{3} + \frac{aq}{2} \right) + x_2^2 \left(F_B a + \frac{F_C}{2} a + \frac{3aq}{4} \right) + x_2 \left(F_B a^2 + \frac{a^2 q}{2} \right) \right]_0^a$$

$$0 = \frac{F_B a^3}{3} + \frac{a^4 q}{8} + \frac{4 F_B a^3}{3} + \frac{5 F_C a^3}{3} + \frac{15 q a^4}{8}$$

$$0 = F_B \cdot \left(\frac{1}{3} + \frac{4}{3} \right) + \frac{5}{3} F_C + qa \left(\frac{1}{8} + \frac{15}{8} \right)$$

$$F_B = \frac{3}{8} \left[-\frac{5}{3} F_C - 2qa \right] = -\frac{5}{16} F_C - \frac{3}{4} qa$$

d) deflection at the center of the cantilever

$$H_1 = -q \frac{x_1^2}{2} + \frac{3}{8} \left[\frac{5}{3} F_C + 2qa \right] x_1 = -q \frac{x_1^2}{2} + \frac{3}{8} 2qa x_1 + \frac{5}{16} F_C x_1 = -\frac{q}{2} \left(x_1^2 - \frac{3}{2} a x_1 \right) + \frac{5}{16} F_C x_1$$

$$H_2 = -q \frac{(x_2+a)^2}{2} + \frac{3}{8} \left[\frac{5}{3} F_C + 2qa \right] (x_2+a) - F_C x_2 = -\frac{q}{2} \left((x_2+a)^2 - \frac{3}{2} a (x_2+a) \right)$$

$$+ F_C \left(\frac{5}{16} (x_2+a) - x_2 \right) = -\frac{q}{2} \left[(x_2+a)^2 - \frac{3}{2} a (x_2+a) \right] + F_C \frac{1}{16} (-11x_2 + 5a)$$

$$w_C = \frac{\partial U^*}{\partial F_C} \Big|_{F_C=0} = \frac{1}{EJ} \left\{ \int_0^a H_1 \frac{\partial H_1}{\partial F_C} dx_1 + \int_0^a H_2 \frac{\partial H_2}{\partial F_C} dx_2 \right\}_{F_C=0}$$

Variational Methods: Examples

$$\begin{aligned} &= \frac{1}{EJ} \left\{ \int_0^a -\frac{q}{2} \left(x_1^2 + \frac{3}{2} a x_1 \right) \cdot \frac{5}{16} x_1 dx_1 + \int_0^a -\frac{q}{2} \left[(x_2+a)^2 - \frac{3}{2} a (x_2+a) \right] \cdot \frac{1}{16} (-11x_2+5a) dx_2 \right\} \\ &= \frac{1}{EJ} \left[\frac{q}{8} \left(\frac{5}{8} a x_1^3 - \frac{5}{16} x_1^4 + \frac{5}{8} a^2 x_2 - a^2 x_2^2 + \frac{1}{24} a x_2^3 + \frac{11}{24} x_2^4 \right) \right]_0^a = \underline{\underline{\frac{1}{EJ} \frac{a^4 q}{12}}} \end{aligned}$$

Thank you!