The Fourier transform of the Heaviside function: a tragedy

Let

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{1}$$

This function is the *unit step* or *Heaviside*[1] function. A basic fact about $H(t)$ is that it is an antiderivative of the Dirac delta function:[2]

$$H'(t) = \delta(t). \tag{2}$$

If we attempt to take the Fourier transform of $H(t)$ directly we get the following statement:

$$\tilde{H}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega t}\, dt = \lim_{B\to+\infty} \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-i\omega B}}{i\omega}.$$

The limit on the right, and the integral itself, does not exist because $\lim_{B\to\infty} e^{-i\omega B}$ does not exist. One might propose that the *average* value of $e^{-i\omega B}$, for fixed ω and $B \to \infty$, is zero. It might then make sense to conclude:

$$\tilde{H}(\omega) \stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}. \tag{3}$$

Certainly this formula is a good candidate for $\tilde{H}$, but it would be nice to get some kind of confirmation because we are clearly on shaky ground here.

Another possibility for accessing $\tilde{H}(\omega)$ is to define[3]

$$H_L(t) = \begin{cases} 1, & 0 < t < L, \\ 0, & t < 0 \text{ or } t > L \end{cases}$$

and consider the Fourier transform:

$$\tilde{H}_L(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^L e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-i\omega L}}{i\omega} = \sqrt{\frac{2}{\pi}}\, e^{-i\omega L/2} \frac{\sin(\omega L/2)}{\omega}.$$

This integral and transform make sense because L is finite. But this calculation doesn't help us. We still have the same difficulty of limits as $L \to \infty$. In fact, we have merely replaced "B" with "L" in the argument which led to (3).

Let's consider less direct routes to get $\tilde{H}(\omega)$. From equation (2) we know, by easy and uncontroversial calculation, that $\tilde{\delta}(\omega) = 1/\sqrt{2\pi}$. But then, from $\mathcal{F}[f'] = i\omega \tilde{f}(\omega)$,

$$\tilde{H}(\omega) \stackrel{?}{=} \frac{1}{i\omega} \mathcal{F}[H'] = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}. \tag{4}$$

This seems to confirm (3); we are getting somewhere.

For further confirmation we form a different approximation to $H(t)$ and take the limit. Let

$$H_\alpha(t) = \begin{cases} e^{-\alpha t}, & t > 0, \\ 0, & t < 0. \end{cases}$$

[1] From *Wikipedia*: Oliver Heaviside (1850–1925) was a self-taught English engineer, mathematician and physicist who adapted complex numbers to the study of electrical circuits, developed techniques for applying Laplace transforms to the solution of differential equations, ... [and] recast Maxwell's mathematical analysis from its original quaternion form to its modern vector terminology, thereby reducing the original twenty equations in twenty unknowns down to the four differential equations in four unknowns we now know as Maxwell's equations.

[2] This statement is precise if $H(t)$ and $\delta(t)$ are regarded as *generalized functions*, also known as *distributions*.

[3] This is what I was doing in class. I made a mess of it.

Note that

$$\lim_{\alpha\to 0^+} H_\alpha(t) = H(t)$$

for every $t \in \mathbb{R}$. Furthermore, if $\alpha > 0$ then the Fourier transform exists:

$$\tilde{H}_\alpha(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\alpha+i\omega)t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + i\omega}.$$

Thus we see the same formula for $\tilde{H}$ appear in the $\alpha \to 0^+$ limit:[4]

$$\tilde{H}(\omega) \overset{?}{=} \lim_{\alpha\to 0^+} \tilde{H}_\alpha(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}. \tag{5}$$

Unfortunately, formula (3) is wrong. Note the formula is undefined at zero. In some sense, as I now show, we have the value at $\omega = 0$ wrong.

Formula (3) doesn't stand up to applying the inverse transform to get back to $H(t)$. Indeed,

$$\begin{aligned} \mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}\right] &= \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{e^{i\omega t}}{\omega}\, d\omega \overset{*}{=} \frac{1}{2\pi i} \left(\int_{-\infty}^0 \cdots + \int_0^\infty \cdots \right) \\ &\overset{**}{=} \frac{1}{2\pi i} \int_0^\infty \frac{e^{i\omega t} - e^{-i\omega t}}{\omega}\, d\omega = \frac{1}{\pi} \int_0^\infty \frac{\sin \omega t}{\omega}\, d\omega = \frac{1}{2}. \end{aligned}$$

There are several comments to make on the above calculation; it is correct with certain caveats. First, the splitting of the integral *at zero* in step $*$ is critical. That is, the two integrals $\int_{-\infty}^0 \cdots$ and $\int_0^\infty \cdots$ give real parts of ∞ and $-\infty$, respectively, but once we change variables and combine the integrals (step $**$) we get cancelation of these real parts and a finite purely imaginary value is revealed. This splitting of an integral at zero is called *computing the Cauchy principle value* of the integral. Another comment is that our calculation is only correct *if* $t > 0$. If $t < 0$ the step $**$ gives an extra negative; the result is $-1/2$. Finally, the integral

$$\int_0^\infty \frac{\sin \omega}{\omega}\, d\omega = \frac{\pi}{2}$$

is by no means trivial. There is no antiderivative of $\operatorname{sinc} x = \sin x / x$ expressible in elementary functions, but contour integral techniques (chapter 20) give the value here.

Thus we have

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}\right](t) = \begin{cases} 1/2, & t > 0, \\ -1/2, & t < 0. \end{cases}$$

This is not $H(t)$ but rather $H(t) - 1/2$. Thus

$$\tilde{H}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} + \frac{1}{2}\sqrt{2\pi}\delta(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} + \sqrt{\frac{\pi}{2}}\, \delta(\omega) \tag{CORRECT!}$$

because $\mathcal{F}^{-1}[\delta(\omega)](t) = 1/\sqrt{2\pi}$.

The only difference between this correct result and our previous calculations (3), (4), and (5) is the value at $\omega = 0$. In fact, it is clear, at least in retrospect, that those calculations *did not apply* to the value at $\omega = 0$.

Exercise/Application (optional). Suppose we know the transform $\tilde{f}(\omega)$ of $f(t)$. Let $g(t) = H(t)f(t)$, so $g(t)$ is the "same signal as $f(t)$ but turned off until time zero." Show that

$$\tilde{g}(\omega) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\tilde{f}(\omega - x)}{x}\, dx + \frac{1}{2}\tilde{f}(\omega).$$

Hint: $\mathcal{F}[u(t)v(t)] = (1/\sqrt{2\pi})\tilde{u}(\omega) * \tilde{v}(\omega)$.

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[4] In this argument we come close to the *Laplace* transform of $H(t)$. The following transform is easy and uncontroversial: $\mathcal{L}[H(t - t_0)] = e^{-st_0}/s$.