Continuum mechanics

Lecture 4

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Generalized Hook's Law

A material body is said to be *ideally elastic* when the body recovers (under isothermal conditions) its original form completely upon removal of the forces causing deformation and there is a one-to-one relationship between the state of stress and state of strain. The generalized Hook's law relates the nine components of stress to the nine components of strain by the linear relation

$$\sigma_{ij} = c_{ijkl} e_{kl},$$

where e_{kl} are the infinitesimal strain components, σ_{ij} are Cauchy stress components and c_{ijkl} are the material parameters.

Generalized Hook's Law

A material is said to be *homogeneous* if the parameters c_{ijkl} do not vary from point to point in the body. The nine equations in

$$\sigma_{ij} = c_{ijkl} e_{kl},$$

contain 81 parameters. Due to the symmetry of both σ_{ij} and e_{kl}, it follows that

$$c_{ijkl} = c_{jikl}, \quad c_{ijkl} = c_{ijlk}$$

and there are only 36 constants.

Generalized Hook's Law

The previous equations can also be expressed in the matrix form

$$\sigma_i = c_{ij} e_j,$$

where

$$\begin{array}{lll} \sigma_{11} = \sigma_1, & \sigma_{22} = \sigma_2, & \sigma_{33} = \sigma_3, \\ \sigma_{12} = \sigma_6, & \sigma_{13} = \sigma_5, & \sigma_{12} = \sigma_4, \end{array}$$

$$\begin{array}{ccc} e_{11} = e_1, & e_{22} = e_2, & e_{33} = e_3, \\ 2e_{12} = e_6, & 2e_{13} = e_5, & 2e_{12} = e_4. \end{array}$$

Note that σ_i and e_j do not constitute the components of tensor. Therefore one cannot transform the components σ_i and e_i like vectors, but these must be rewritten to the tensor components σ_{ij} or e_{kl}, transformed and consequently written to the vector form. The same rule is valid for matrix components c_{ij}.

Strain Energy Density Function

In ideal elasticity, when η is differentiable and $d\eta = 0$, it is assumed that all of the input work is converted into internal energy in the form of recoverable stored elastic energy. If we denote the strain energy per unit volume by $U_0 = U_0(e_{ij})$, then from

$$\sigma_{ij} de_{ij} + \rho T d\eta = dU_0.$$

follows

$$\sigma_{ij} = \frac{\partial U_0}{\partial e_{ij}} \quad \text{for } \eta = const.$$

Strain Energy Density Function

When temperature effects are involved, the strain energy density is a function of strains and temperature $U_0 = U_0(e_{ij}, T)$ and it must be introduced the *free-energy function* Ψ combining the strain energy density U_0 and entropy η to be treated as the potential

$$\rho\Psi = U_0 - \rho T\eta.$$

Since Ψ is a function of the strains e_{ij} and temperature T, we have

$$dU_0 = \rho\frac{\partial\Psi}{\partial e_{ij}}de_{ij} + \left(\rho\frac{\partial\Psi}{\partial T} + \rho\eta\right)dT + \rho T d\eta.$$

Strain Energy Density Function

Substituting

$$\sigma_{ij} de_{ij} + \rho T d\eta = dU_0.$$

into

$$dU_0 = \rho \frac{\partial \Psi}{\partial e_{ij}} de_{ij} + \left(\rho \frac{\partial \Psi}{\partial T} + \rho \eta \right) dT + \rho T d\eta$$

yields

$$0 = \left(\rho \frac{\partial \Psi}{\partial e_{ij}} - \sigma_{ij} \right) de_{ij} + \rho \left(\frac{\partial \Psi}{\partial T} + \eta \right) dT.$$

Because the differentials de_{ij} and dT are independent to each other, the previous equation leads to the relations

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial e_{ij}}, \quad \eta = \rho \frac{\partial \Psi}{\partial T}.$$

Strain Energy Density Function

From

$$\sigma_{ij} = \frac{\partial U_0}{\partial e_{ij}} \quad \text{for } \eta = const$$

and

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial e_{ij}}, \quad \eta = \rho \frac{\partial \Psi}{\partial T}$$

also follows important result, if the deformation process is isothermal and reversible, i.e. the temperature T is constant during the very slow quasi-static process in the thermodynamic equilibrium, then

$$\sigma_{ij} = \rho \left. \frac{\partial \Psi}{\partial e_{ij}} \right|_{T=const} = \left. \frac{\partial U_0}{\partial e_{ij}} \right|_{\eta=const} .$$

The first relation can be used to reduce the number of independent elastic constants to 21.

Strain Energy Density Function

The substitution of the generalized Hook's law

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

into

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial e_{ij}}$$

gives

$$\rho \frac{\partial \Psi}{\partial e_{ij}} = c_{ijkl} e_{kl}.$$

The partial differentiation of this equation with respect to e_{kl} gives

$$\rho \frac{\partial}{\partial e_{kl}} \left(\frac{\partial \Psi}{\partial e_{ij}} \right) = c_{ijkl}.$$

The interchanging the indices kl with ij we obtain

$$\rho \frac{\partial}{\partial e_{ij}} \left(\frac{\partial \Psi}{\partial e_{kl}} \right) = c_{klij}.$$

Strain Energy Density Function

Since the order of the previous partial differentiation is unimportant, it follows that

$$c_{ijkl} = c_{klij}.$$

The importance of this result is that it was obtained from the thermodynamics considerations without any symmetry requirements to the stress or strains tensors σ_{ij} or e_{ij}, respectively. Consequently the array c_{ij} in

$$\sigma_i = c_{ij} e_j$$

is symmetric and the number of independent constants is reduced from 36 to 21.

Strain Energy Density Function

It is also useful to introduce *thermodynamic potential* Ψ^* as follows

$$\rho\Psi^* = \rho\Psi - \sigma_{ij}e_{ij},$$

for which yields

$$\rho d\Psi^* = -\rho\eta dT - e_{ij}d\sigma_{ij}.$$

Analogically to the previous derivations, the strain components e_{ij} can be evaluate for the isothermal deformations process as

$$e_{ij} = -\rho\frac{\partial\Psi^*}{\partial\sigma_{ij}} \quad \text{for } T = const.$$

Elastic Symmetry

When c_{ij} do not depend on the orientation, the material is *isotropic* for which the stress-strain relationship is of the form

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk},$$

where μ and λ are called *Lamé constants* related to *the shear modulus* G, *Young's modulus* E and *Poisson's ratio* ν by

$$\mu = G = \frac{E}{2(1+\nu)}, \quad \mu = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

The coefficients c_{ij} for isotropic materials are

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu, \quad c_{12} = c_{13} = c_{23} = \lambda,$$

$$c_{44} = c_{55} = c_{66} = \mu$$

and all other c_{ij}'s are zero.

Elastic Symmetry

Whenever three mutually orthogonal planes of elastic symmetry for a material exist, the material is said to be *orthotropic*

$$c_{11} = E_1 \frac{1-\nu_{23}\nu_{32}}{\Delta}, \quad c_{12} = E_1 \frac{\nu_{21}+\nu_{31}\nu_{23}}{\Delta} = E_2 \frac{\nu_{12}+\nu_{32}\nu_{13}}{\Delta},$$

$$c_{13} = E_1 \frac{\nu_{31}+\nu_{21}\nu_{32}}{\Delta} = E_3 \frac{\nu_{13}+\nu_{12}\nu_{23}}{\Delta}, \quad c_{22} = E_2 \frac{1-\nu_{13}\nu_{31}}{\Delta},$$

$$c_{23} = E_2 \frac{\nu_{32}+\nu_{12}\nu_{31}}{\Delta} = E_3 \frac{\nu_{23}+\nu_{21}\nu_{13}}{\Delta}, \quad c_{22} = E_3 \frac{1-\nu_{12}\nu_{21}}{\Delta},$$

$$c_{44} = G_{23}, \quad c_{55} = G_{31}, \quad c_{66} = G_{12}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13},$$

where E_1, E_2, E_3 are Young's moduli in 1, 2 and 3 directions, $\nu_{ij} = -e_{jj}/e_{ii}$ are Poisson's ratios of transverse strain in the j-direction to the axial strain in the i-direction, when stressed in the i-direction and G_{23}, G_{13}, G_{12} are shear moduli in the 2-3, 1-3 and 1-2 planes.

Elastic Symmetry

More convenient it is used the components of the compliance matrix $\boldsymbol{S}$, which is inverse to the stiffness matrix $c_{ij} = \boldsymbol{C}$ and obeys the relation

$$e_i = S_{ij}\sigma_j$$

or in tensor form

$$e_{ij} = S_{ijkl}\sigma_{kl},$$

where

$$\boldsymbol{S} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}.$$

Elastic Symmetry

It is of interest in the study of laminated plates of orthotropic layers to have an explicit form of the transformation equations relating the elastic moduli in one coordinate system to those in another coordinate system. The parameters c_{ijkl} are components of a fourth-order tensor obeying the transformation rule

$$c'_{ijkl} = a_{im} a_{jn} a_{kp} a_{lq} c_{mnpq},$$

where a_{im} denotes the direction cosines associated with the x'_i-axis and the x_m-axis

$$a_{im} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_m.$$

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

A change in the temperature of a body leads to its deformation due to the thermal expansion, even if there is no external load acting on it. When the thermal expansion are prevented by boundary conditions or other constrains, the body develops also the thermal stresses in addition to stresses caused by other loads. In this case, the temperature T, measured above an arbitrary reference temperature T_0, enters as a parameter in the linear terms of strains e_{ij} of the free energy Ψ as follows

$$\rho\Psi = \frac{1}{2}c_{ijkl}e_{ij}e_{kl} - \beta_{ij}(T - T_0)e_{ij}.$$

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

The linear thermoelastic constitutive equations and entropy are then given by

$$\sigma_{ij} = \rho \frac{\partial \Psi}{\partial e_{ij}} = c_{ijkl} e_{kl} - \beta_{ij} (T - T_0),$$

and

$$\eta = \rho \frac{\partial \Psi}{\partial T} = -\beta_{ij} e_{ij},$$

where β_{ij} are the Cartesian components of the tensor $\boldsymbol{\beta}$ representing the material thermal expansion parameters.

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

For an isotropic body the previous equations take the simple forms

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - \beta \delta_{ij} (T - T_0), \quad \eta = -\beta e_{kk},$$

where

$$\beta = (2\mu + 3\lambda)\alpha$$

and α is the linear coefficient of thermal expansion. Setting the stresses on the left-hand side in the first expression for stresses σ_{ij} equal to zero, then it is obtained

$$e_{kk} = \alpha(T - T_0).$$

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

The isothermal deformation process of the body will be considered in the following elasticity problems and applied variational methods. It means, that the temperature of the deformed body during its quasi-static deformation process must be constant, i.e. $T = T_0$. Then the equation

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - \beta \delta_{ij} (T - T_0)$$

is reduced to the standard constitutive relation for isotropic material

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}.$$

As the consequence, the material parameters λ and μ are known as isothermal.

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

On the other hand, the relations

$$\sigma_{ij} = \rho \left. \frac{\partial \Psi}{\partial e_{ij}} \right|_{T=const} = \left. \frac{\partial U_0}{\partial e_{ij}} \right|_{\eta=const}.$$

show that the stress tensor σ_{ij} in an isothermal process can be evaluated as the derivative of the deformation energy U_0 with respect to the deformations, if the entropy η is constant. The expression for the entropy

$$\eta = -\beta e_{kk}$$

is constant only if

$$T - T_0 \sim e_{kk}.$$

This condition is equivalent to the equation for the deformations during the isothermal process and zero stresses σ_{ij}.

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

The substitution of temperature

$$T - T_0 \sim e_{kk}$$

into the free energy $\rho\Psi$, it is obtained the strain energy density U_0, which can be written in the form

$$U_0 = \frac{1}{2} c^{ad}_{ijkl} e_{ij} e_{kl},$$

where $c^{ad}_{ijkl} = c^{ad}_{ijkl}(T)$ are *the adiabatic elastic parameters* linearly depending on the temperature T. Supposing the isotropic material, it is obtained the constitutive relation

$$\sigma_{ij} = \frac{\partial U_0}{\partial e_i j} = \lambda^{ad} \delta_{ij} e_{kk} + 2\mu e_{ij}.$$

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

In practice, the dependence of $c_{ijkl}^{ad} = c_{ijkl}^{ad}(T)$ on the temperature T is very weak and we can assume that the difference between c_{ijkl}^{ad} and c_{ijkl} can be neglected. This assumption allows us to handle the thermoelastic constitutive problem in an elegant mathematical way. We will therefore assume a quasi-static loading process, i.e. an isothermal process, that is always in thermodynamic equilibrium.

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

However, the stress tensor components σ_{ij} are calculated from the strain energy density U_0 under the condition of constant entropy η, i.e. the adiabatic thermodynamic process, according to the equation

$$\sigma_{ij} = \frac{\partial U_0}{\partial e_{ij}},$$

where the unnecessary notation $\eta = const$ is omitted, with additional deformation condition

$$e_{kk} = \alpha(T - T_0).$$

It will be supposed that the strain energy density U_0 is independent of the temperature T, it is conservative, differentiable and reversible.

Deformations Due to Temperature Change and Thermoelastic Constitutive Equations

Similarly, the so-called *complementary strain energy* U_0^* can be introduced to the evaluation of the strains e_{ij} in the isothermal process using the thermodynamic potential Ψ^* defined above as

$$\rho\Psi^* = \rho\Psi - \sigma_{ij}e_{ij}.$$

Analogously, the complementary strain energy U_0^* is defined as follows

$$U_0^* = U_0 - \sigma_{ij}e_{ij}$$

and according to the previous assumptions and derivations, we will calculate the strains e_{ij} in the isothermal process as follows

$$e_{ij} = \frac{\partial U_0^*}{\partial \sigma_{ij}} + \alpha(T - T_0)\delta_{ij}.$$

Thank you!